Noncommutative Maslov Index and Eta-Forms

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ABSTRACT. We define and prove a noncommutative generalization of a formula relating the Maslov index of a triple of Lagrangian subspaces of a symplectic vector space to eta-invariants associated to a pair of Lagrangian subspaces. The noncommutative Maslov index, defined for modules over a C^*-algebra $\mathcal{A}$, is an element in $K_0(\mathcal{A})$. The generalized formula calculates its Chern character in the de Rham homology of certain dense subalgebras of $\mathcal{A}$. The proof is a noncommutative Atiyah-Patodi-Singer index theorem for a particular Dirac operator twisted by an $\mathcal{A}$-vector bundle. We develop an analytic framework for this type of index problem.

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Introduction

The purpose of this paper is twofold. We establish a noncommutative generalization of a formula by Cappell, Lee and Miller [**CLM**] relating the Maslov index of a triple of Lagrangian subspaces of a symplectic vector space to certain η-invariants which can be associated to a pair of Lagrangian subspaces. The formula was generalized to families of Lagrangian subspaces by Bunke and Koch [**BK**]. Our Maslov index will be defined for C^*-modules and will be an element in the K-theory of the C^*-algebra, whereas the η-invariants are replaced by noncommutative differential forms. The proof of our formula is a noncommutative index theorem for a particular Dirac operator twisted by a C^*-vector bundle on a two-dimensional manifold with boundary and cylindric ends. The index theorem calculates the Chern character of index of the Dirac operator in the de Rham homology of certain dense subalgebras of the C^*-algebra.

The second aim is to provide a precise analytic framework for the index theory of Dirac operators over a certain type of dense subalgebras of C^*-algebras based on heat kernel methods and the Quillen superconnection formalism. In particular these dense subalgebras are assumed to be projective limits of Banach algebras, so that we mostly deal with Banach space valued functions. Our results apply to higher index theory for invariant Dirac operators on covering spaces. Proofs in higher index theory based on the superconnection formalism have been given before [**Lo1**] [**Lo2**] [**LP1**] [**LP2**] [**Wu**]. Some analytical methods relevant for the general situation have been developed by Lott in [**Lo3**].

The Maslov index $\tau(L_0, L_1, L_2)$ of a triple of Lagrangian subspaces (L_0, L_1, L_2) of $\mathbb{R}^{2n}$ endowed with the standard symplectic form ω is defined as the signature of the quadratic form q on $L_0 \cap (L_1 + L_2)$ [**LV**] given by

$$q(x_1 + x_2) := \omega(x_2, x_1), \ x_i \in L_i \ .$$

Its geometric significance comes from a gluing formula for signatures of manifolds with boundary [**Wa**]: Let M^{4n} be an oriented manifold with boundary and let N be a hypersurface in M with boundary. Cutting along N produces two (topological) manifolds M_1, M_2 with boundary. The images of $H^{2n-1}(N)$ and $H^{2n-1}(M_i)$, $i = 1, 2$, in $H^{2n-1}(\partial N)$ are Lagrangian subspaces with respect to the symplectic form induced by the cup product. Up to sign the difference of the signatures $\sigma(M) - \sigma(M_1) - \sigma(M_2)$ equals the Maslov index of these subspaces.

The η-invariant associated to a pair (L_0, L_1) of Lagrangian subspaces is defined as the η-invariant of the operator $D_I = I_0 \frac{d}{dx}$ on $L^2([0,1], \mathbb{R}^{2n})$ with boundary conditions $f(0) \in L_0$, $f(1) \in L_1$. Here I_0 is the skewsymmetric matrix representing

the symplectic form ω with respect to the standard scalar product on $\mathbb{R}^{2n}$. Then

$$\eta(L_0, L_1) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \text{Tr} \ D_I e^{-tD_I^2} dt \ .$$

It is a regularization of the difference between the number of positive and negative eigenvalues of D_I. The η-invariant $\eta(L_0, L_1)$ occurs as a correction term in a gluing theorem for η-invariants [**Bu**]. The formula

$$\tau(L_0, L_1, L_2) = \eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0) \ ,$$

which will be generalized to a noncommutative context, can be interpreted as reflecting the connection between the gluing problems for signatures and η-invariants via the Atiyah-Patodi-Singer index theorem.

The noncommutative Maslov index and the η-forms should play a similar role in gluing problems for higher signatures for manifolds with boundaries [**LP3**], [**LLP**], [**LLK**], higher η-invariants and ρ-invariants [**Lo1**]. In §4.5 we explain how our formula is related to a gluing formula for noncommutative η-forms for Dirac operators on the circle.

The proof by Cappell, Lee and Miller relies on the axiomatic properties of the Maslov index, which are not expected to hold for a noncommutative generalization. Bunke and Koch [**BK**] formulated an index problem whose solution, an Atiyah-Patodi-Singer type index theorem, yields the above formula. They interpreted the Maslov index as the index of a Dirac operator on a two-dimensional manifold with boundary and six cylindric ends isometric to $\mathbb{R}^+ \times [0, 1]$. The Lagrangian subspaces enter in the definition of the boundary conditions. The purpose of their approach was to prove a formula for families of Lagrangian subspaces.

Our proof is a generalization of the approach of Bunke and Koch. Instead of a family of Dirac operators we consider a Dirac operator twisted by a C^*-vector bundle [**MF**].

Before stating the result we give a short introduction in the type of noncommutative index theorem considered in this paper:

Family index theorems describe Fredholm operators depending continuously on a parameter from some compact space. The index of a family of operators is an element in K-theory of the base space. If the kernel and the cokernel are vector bundles, then the index is the difference of the classes of these bundles.

One may reformulate this situation by replacing the base space B by the C^*-algebra $C(B)$ and the family of operators by an operator on a $C(B)$-module. The index is then in the C^*-algebraic K-theory $K_0(C(B))$, which is naturally isomorphic to $K^0(B)$.

Miščenko and Fomenko [**MF**] formalized and generalized this point of view by defining Fredholm operators on Hilbert C^*-modules for general C^*-algebras. The index of such a Fredholm operator is an element in the K-theory of the C^*-algebra. They also elaborated a theory of pseudodifferential operators over C^*-algebras. Important examples for geometric applications are Dirac operators associated to C^*-vector bundles.

In order to formulate a noncommutative analogue of the family index theorem, which calculates the Chern character of the index of a family of Dirac operators in the cohomology of the base space, now assumed to be a manifold, analogues of differential forms, de Rham cohomology and the Chern character are

needed. Karoubi [**Ka**] introduced a complex of differential forms $(\hat{\Omega}_* \mathcal{A}, \mathrm{d})$ associated to a Fréchet algebra $\mathcal{A}$, its de Rham homology $H_*^{dR}(\mathcal{A})$ and a Chern character $\mathrm{ch} : K_0(\mathcal{A}) \to H_*^{dR}(\mathcal{A})$, which is defined if in addition $\mathcal{A}$ is a local Banach algebra. Unfortunately the de Rham homology of a C^*-algebra does not behave well, in particular the de Rham homology of a commutative unital C^*-algebra is in general not the cohomology of the corresponding compact space. By considering the de Rham homology of the algebra $C^\infty(B)$ instead of $C(B)$ one recovers the Chern character from differential geometry. This can be interpreted as reflecting the fact that a differentiable structure on B is needed for the definition of de Rham cohomology.

In order to get a reasonable Chern character in the general situation one chooses a dense subalgebra $\mathcal{A}_\infty$ holomorphically closed under the functional calculus in $\mathcal{A}$. Then $K_0(\mathcal{A}_\infty)$ is canonically isomorphic to $K_0(\mathcal{A})$, and the Chern character yields a homomorphism

$$\mathrm{ch} : K_0(\mathcal{A}) \to H_*^{dR}(\mathcal{A}_\infty) \ .$$

In this setting homological index theorems for Dirac operators associated to C^*-vector bundles can be formulated and – at least formally – a noncommutative version of Bismut's family index theorem makes sense [**BGV**]. Note that formally the noncommutative situation is less complex than the family case since the Riemannian metric is fixed.

For the generalization of the heat kernel theory additional conditions have to be imposed on the algebra $\mathcal{A}_\infty$: It should be the limit of a projective system $\{\mathcal{A}_i\}_{i \in \mathbb{N}_0}$ of involutive Banach algebras with $\mathcal{A}_0 = \mathcal{A}$, such that there are dense embeddings $\mathcal{A}_{i+1} \hookrightarrow \mathcal{A}_i$ and such that each $\mathcal{A}_i$ is closed with respect to the holomorphic functional calculus in $\mathcal{A}$. The motivating example is $\mathcal{A}_i := C^i(B)$ for a closed manifold B. More generally, if δ is an involutive closed derivation on $\mathcal{A}$ with $\cap_i \mathrm{dom}\, \delta^i$ dense in $\mathcal{A}$, then the projective system given by $\mathcal{A}_i := \mathrm{dom}\, \delta^i$ with norm

$$\|a\|_i := \sum_{j=0}^{i} \|\delta^j(a)\|$$

fulfills the conditions. For a discrete finitely generated group G the unbounded operator D on $l^2(G)$ defined by $D 1_g = l(g) 1_g$, where l is a word length function on G, induces a closable derivation $\delta(f) = [D, f]$ on $B(l^2(G))$. Furthermore $\mathbb{C}G \subset \mathcal{A}_i := \mathrm{dom}\, \overline{\delta}^i \cap C_r^*(G)$. The projective limit $\mathcal{A}_\infty \subset C_r^*(G)$ is closely related to the algebra employed by Connes and Moscovici in their proof of the Novikov conjecture for Gromov hyperbolic groups via higher index theory [**CM**]. Using this setting our results can be applied to higher index theory. A proof of a higher Atiyah-Singer index theorem based on heat kernel methods was given by Lott [**Lo2**] and a higher Atiyah-Patodi-Singer index theorem was proved by Leichtnam and Piazza [**LP1**] [**LP2**]. Lott introduced higher η-invariants and ρ-invariants [**Lo1**]. A motivation for higher index theory is the study of higher signatures and the Novikov conjecture. Index theory for Dirac operators over C^*-algebras in general has been applied to the study of manifolds with positive scalar curvature [**PS**] [**Ros2**].

The proofs of the higher index theorems mentioned above rely on the comparison with the situation on the covering space where one can deal with operators on sections of complex vector bundles. In the general setting the main difficulty lies in the fact that the calculus of regular operators on a Hilbert C^*-module and the calculus of pseudodifferential operators over a C^*-algebra are not sufficient for the study of the heat semigroup since we have to deal with vector bundles whose fibers

are projective $\mathcal{A}_\infty$-modules. By proving that the de Rham homology behaves well under the projective limit we justify the fact that one can deal with Banach spaces instead of Fréchet spaces. Then the theory of holomorphic semigroups can be used instead of the calculus of selfadjoint operators. We define appropriate operator spaces on L^2-spaces of vector valued functions, for example Hilbert-Schmidt operators and trace class operators. Duhamel's principle is used for the construction and the short time asymptotics of the heat kernels. The long time asymptotics is more intricate: here we have to get hold of the spectrum of the Dirac operator. We adapt a method developed by Lott, namely a restricted pseudodifferential operator calculus giving information about the resolvent set of the Dirac operator and the regularizing properties of the resolvents [**Lo3**].

Although the theory is developed only for a particular two-dimensional manifold and a trivial vector bundle, the relevant parts generalize to the Atiyah-Patodi-Singer index problem for Dirac operators over C^*-algebras, which will be considered elsewhere.

Now we can formulate the noncommutative version of the equation relating the Maslov index and the η-invariants.

Let $\mathbb{C}^{2n}$ be endowed with the skewhermitian form ω induced by the standard symplectic form on $\mathbb{R}^{2n}$ via the identification $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$.

For a C^*-algebra $\mathcal{A}$ a Lagrangian projection on $\mathcal{A}^{2n}$ is a selfadjoint projection $P \in M_{2n}(\mathcal{A})$ fulfilling $PI_0 = (1 - I_0)P$. Two projections are called transverse if their sum is invertible.

As above one associates an $\mathcal{A}$-valued hermitian form q to every triple $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of pairwise transverse Lagrangian projections. The class $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) := [q] \in K_0(\mathcal{A})$ is called Maslov index of $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

An η-form $\eta(\mathcal{P}_0, \mathcal{P}_1) \in \hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s}$ can be associated to a pair of transverse Lagrangian projections $(\mathcal{P}_0, \mathcal{P}_1)$ with $\mathcal{P}_i \in M_{2n}(\mathcal{A}_\infty)$, $i = 0, 1$. Here $[\ ,\ ]_s$ denotes the supercommutator. The η-form is defined via a superconnection associated to the operator $D_I = I_0 \frac{d}{dx}$ on $L^2([0,1], \mathcal{A}^{2n})$ with boundary conditions $f(0) = \mathcal{P}_0 f(0)$ and $f(1) = \mathcal{P}_1 f(1)$.

Then our main result is:

THEOREM. *For a triple $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of pairwise transverse Lagrangian projections with $\mathcal{P}_i \in M_{2n}(\mathcal{A}_\infty)$, $i = 0, 1, 2$,*

$$\operatorname{ch} \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [\eta(\mathcal{P}_0, \mathcal{P}_1) + \eta(\mathcal{P}_1, \mathcal{P}_2) + \eta(\mathcal{P}_2, \mathcal{P}_0)] \in H_*^{dR}(\mathcal{A}_\infty) \ .$$

This paper is based on the author's PhD thesis. I would like to thank my supervisor Ulrich Bunke for drawing my attention to the problem and for fruitful discussions, and Margit Rösler for the introduction into the theory of semigroups.

Summary

The paper is organized in the following way:

As mentioned above, the Maslov index $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ is the index of a Dirac operator D^+ twisted by a C^*-vector bundle on a two-dimensional spin manifold M with six cylindric ends isometric to $[0, \infty) \times [0, 1]$.

In Chapter 1 the manifold M is described and we explain in more detail the family index theorem of Bunke and Koch [**BK**]. Then the universal differential algebra $\hat{\Omega}_* \mathcal{A}_\infty$, the de Rham homology $H_*^{dR}(\mathcal{A}_\infty)$ and the Chern character are introduced and investigated. It is shown that $H_*^{dR}(\mathcal{A}_\infty)$ is the projective limit of $H_*^{dR}(\mathcal{A}_i)$. Moreover Lagrangian projections and the Maslov index are defined.

In Chapter 2 we introduce the Dirac operator $D = D^+ \oplus D^-$ on M whose boundary conditions are defined by a triple of Lagrangian projections $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$. Furthermore the operator D_I is defined and its properties on the Hilbert $\mathcal{A}$-module $L^2([0,1], \mathcal{A}^{2n})$ are studied. Then we show that D^+ is Fredholm between appropriate Hilbert C^*-modules and that its index equals $\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$. We define a compact perturbation $D(\rho)$ of D such that $D(\rho)^+$ is surjective. Then the index of D^+ can be expressed in terms of the kernel of $D(\rho)$.

Chapter 3 is devoted to heat semigroups and their integral kernels, in particular those associated to D_I and $D(\rho)$.

In Chapter 4 we introduce superconnections in order to define the η-form. Now we can formulate the index theorem. The remainder of the chapter is devoted to its proof. We introduce the rescaled superconnection $A(\rho)_t$ associated to $D(\rho)$ and study the family of operators $e^{-A(\rho)_t^2}$. We show that it is a family of integral operators with smooth integral kernel and obtain estimates for the integral kernel for small t. Once the heat kernel theory is established, the proof of the index theorem itself is fairly standard. We follow the proof in [**BK**], which is modelled on Melrose's b-calculus [**Me**], and compare the limit of a generalized supertrace of $e^{-A(\rho)_t^2}$ for $t \to \infty$, which is the Chern character of the index of D^+, with its limit for $t \to 0$. Since the differences between the calculus for Dirac operators associated to complex vector bundles and the one for Dirac operators associated to a projective system of vector bundles as developed here are subtle, the proof is given in detail.

In Chapter 5 the functional analytic framework is developed. The function spaces we deal with are introduced, as for example the L^2-spaces of vector valued functions, and operators on them are studied. In particular we define and study appropriate notions of adjointable operators, Hilbert-Schmidt and trace class operators. Furthermore we recall the properties of Fredholm operators and regular operators on Hilbert C^*-modules, and collect some facts from holomorphic semigroup theory. The reader is advised to go through the definitions of this chapter first in order to get acquainted with the functional analytic setting.

Notation and conventions

If not specified vector spaces and algebras are complex, manifolds are smooth.

We often deal with $\mathbb{Z}/2$-graded spaces. Then $[\,,\,]_s$ denotes the supercommutator and tr_s the supertrace. In a graded context the tensor products are graded. For an ungraded vector space V, we denote by V^+ resp. V^- the same space endowed with a grading: all elements are homogeneous of positive resp. negative degree.

Tensor products denoted by $\otimes$ are completed. The way of completion is indicated by a suffix in all but the two most common cases: In the case of Hilbert C^*-modules $\otimes$ means the Hilbert C^*-module tensor product, and if one of the spaces is a nuclear locally convex space, then $\otimes$ means $\otimes_\pi$ or $\otimes_\varepsilon$. The algebraic tensor product is denoted by $\odot$.

By a differentiable function on an open subset of $[0,1]^n$ we understand a function that can be extended to a differentiable function on an open subset of $\mathbb{R}^n$.

This induces the notion of a differentiable function on a manifold with corners, in our case $M \times M$.

If S, X are sets with $S \subset X$, then the characteristic function of S is denoted by $1_S : X \to \{0, 1\}$. If X is a metric space, $y \in X$ and $S \subset X$ then $d(y, S) := \inf_{x \in S} d(x, y)$. For $S_1, S_2 \subset X$ we set $d(S_1, S_2) := \inf_{x \in S_1} d(x, S_2)$.

If E_i, $i = 1, 2$, is a vector bundle on a space X_i, $i = 1, 2$, and $p_i : X_1 \times X_2 \to X_i$ is the projection, then $E_1 \boxtimes E_2 = p_1^* E_1 \otimes p_2^* E_2$ on $X_1 \times X_2$.

We use the notions from [**BGV**] in the context of spin geometry. In addition let a Dirac bundle be a selfadjoint Clifford module endowed with a Clifford connection with respect to which the metric is parallel. Our sign conventions differ from those in [**BGV**] since we deal with right modules over the algebra of differential forms.

The value of the constant C used in estimates may vary during a series of estimates without an explicit remark.

CHAPTER 1

Preliminaries

1.1. The geometric situation

In this section the two-dimensional spin manifold M with boundary and cylindric ends, a Dirac bundle E on it and the associated Dirac operator will be introduced. The open covering $\mathcal{U}(r, b)$ of M constructed in this section will be used for cutting and pasting arguments later on. Furthermore we will fix a flat open set $F \subset M$ containing the boundary and the cylindric ends and trivializations of $TM|_F$ and of $E|_F$, which will be used in the definition of the boundary conditions for the Dirac operator in §2.1.1.

We begin by defining the manifold M.

For $k \in \mathbb{Z}/6$ let Z_k be a copy of $\mathbb{R} \times [0, 1]$. With the euclidian metric and the standard orientation Z_k is an oriented Riemannian manifold with boundary. Let (x_1^k, x_2^k) be the euclidian coordinates of Z_k.

For $r \geq -\frac{1}{2}$, $b \leq \frac{1}{3}$ and $k \in \mathbb{Z}/6$ let

$$F_k(r, b) := \{(x_1^k, x_2^k) \in \,]r, \infty[\times [0, 1] \, \cup \,] - 1, r] \times \big([0, b[\cup]1 - b, 1]\big)\} \subset Z_k \ .$$

We define

$$F(r, b) := \Big(\bigcup_{k \in \mathbb{Z}/6} F_k(r, b) \Big) / \sim$$

with $(x_1^k, x_2^k) \sim (-\frac{3}{2} - x_1^{k-1}, 1 - x_2^{k-1})$ for $(x_1^k, x_2^k) \in \,] - 1, -\frac{1}{2}[\times [0, b[$ and $k \in \mathbb{Z}/6$.

Then $F(r, b)$ inherits the structure of an oriented Riemannian manifold from the sets $F_k(r, b)$.

The set $F(-\frac{1}{2}, \frac{1}{3}) \backslash \overline{F(-\frac{1}{3}, \frac{1}{4})}$ is diffeomorphic to the open ring $B_1(0)^\circ \backslash B_{1/2}(0) \subset \mathbb{R}^2$ via an oriented diffeomorphism ϕ. We define the manifold with boundary

$$M := F(-\tfrac{1}{2}, \tfrac{1}{3}) \cup_\phi B_1(0)^\circ \ .$$

For $r > -\frac{1}{2}$, $b \leq \frac{1}{3}$ we identify $F(r, b)$ and $F_k(r, b)$ with the corresponding subsets in M. The sets $F_k(r, b)$ are coordinate patches of M with the coordinates (x_1^k, x_2^k) from above.

Extend the orientation and metric from $F := F(0, \frac{1}{4})$ to the whole of M and endow TM with the Levi-Cività connection. In the following we identify TM and T^*M.
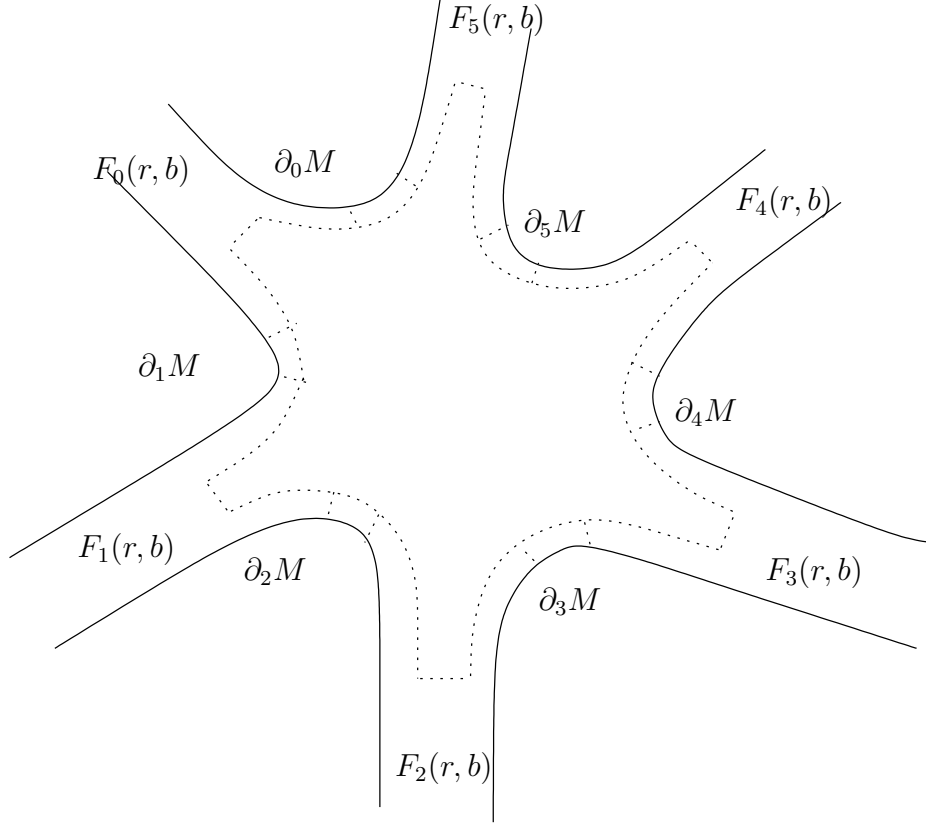
FIGURE 1. The manifold M

For $r \geq 0$ and $b \leq \frac{1}{4}$ we define an open covering $\mathcal{U}(r,b) = \{\mathcal{U}_k\}_{k \in J}$ of M as follows. Let J be the union of $\mathbb{Z}/6$ with a one-element set $\{\clubsuit\}$. For $k \in \mathbb{Z}/6$ let $\mathcal{U}_k := F_k(r,b)$ and let $\mathcal{U}_\clubsuit := M \setminus \overline{F(r+1, b/2)}$.

For $r \geq 0$ let $M_r := M \setminus \overline{F(r,0)}$.

The connected components of ∂M are labelled $\partial_k M$, $k \in \mathbb{Z}/6$, in such a way that $\partial_k M \cap F_k(r,b) \subset \{x_2^k = 0\}$ and $\partial_{k+1} M \cap F_k(r,b) \subset \{x_2^k = 1\}$.

The manifold M can be embedded diffeomorphically into $\mathbb{R}^2$, even with a diffeomorphism that is an isometry on the complement of M_r for some $r > 0$. The image of the embedding is illustrated by figure 1.

Now we define the Dirac operator. Choose a spin structure on M and fix $d \in \mathbb{N}$. Let S be the spinor bundle endowed with the Levi-Cività connection and a parallel hermitian metric. Let E be the graded vector bundle $S \otimes ((\mathbb{C}^+)^d \oplus (\mathbb{C}^-)^d)$. The hermitian metric on S and the standard hermitian product on $\mathbb{C}^d$ induce a hermitian metric $\langle \cdot, \cdot \rangle$ on E.

Furthermore there are connections on E and its dual E^* induced by the Clifford connection on S and the de Rham differential. The Dirac operator associated to the Dirac bundle E is denoted by $\partial\!\!\!/_E$.

The oriented orthonormal frames $((-1)^k dx_1^k, (-1)^k dx_2^k)$ of $TM|_{F_k(0,\frac{1}{4})}$ patch together to an oriented orthonormal frame (e_1, e_2) of $TM|_F$. The Clifford multiplication $c : TM \to \operatorname{End} E$ induces an even parallel endomorphism

$$I = c(e_2)c(e_1) : E|_F \to E|_F$$

defining a skew-hermitian form

$$E|_F \times E|_F \to \mathbb{C}, \ (x, y) \mapsto \langle x, Iy \rangle \ .$$

Since the holonomy of $TM|_F$ is $4\pi\mathbb{Z}$ (measured with respect to any trivialization of TM on the whole of M), there are nonvanishing parallel sections of $S^+|_F$ and $S^-|_F$. We choose a parallel unit section s of $S^+|_F$ and fix once and for all the trivialization of $E|_F$ defined for $x \in F$ by

$$E_x^+ = S_x^+ \otimes (\mathbb{C}^+)^d \oplus S_x^- \otimes (\mathbb{C}^-)^d \to (\mathbb{C}^+)^d \oplus (\mathbb{C}^+)^d \ ,$$

$$(s(x) \otimes v) \oplus (ic(e_1)s(x) \otimes w) \mapsto (v, w) \ ,$$

and

$$E_x^- = S_x^- \otimes (\mathbb{C}^+)^d \oplus S_x^+ \otimes (\mathbb{C}^-)^d \to (\mathbb{C}^-)^d \oplus (\mathbb{C}^-)^d \ ,$$

$$(ic(e_1)s(x) \otimes v) \oplus (s(x) \otimes w) \mapsto (v, w) \ .$$

With respect to this trivialization the endomorphism $I|_{E^+}$ corresponds to

$$I_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in M_{2d}(\mathbb{C})$$

and $I|_{E^-}$ to $-I_0$.

1.2. The family index theorem

In order to give a motivation for the definitions in the subsequent sections we sketch the corresponding family index theorem. The presentation follows [**BK**].

Let $\mathbb{C}^{2d}$ be endowed with the standard hermitian product $\langle \ , \ \rangle$ and the skew-hermitian form $(x, y) \mapsto \langle x, I_0 y \rangle$.

Let B be a compact space and let (L_0, L_1, L_2) be a triple of pairwise transverse Lagrangian subbundles of the trivial bundle $B \times \mathbb{C}^{2d}$. For any $b \in B$ the Lagrangian subspaces $L_i(b) \subset \mathbb{C}^{2d}$ define parallel Lagrangian subbundles of $E^+|_F$ via the trivialization fixed in the previous section.

Let $D^+(b)$ be the Dirac operator associated to E with

$$\operatorname{dom} D^+(b) := \{s \in C_c^\infty(M, E^+) \mid s(x) \in L_i(b) \text{ for } x \in \partial_i M \cup \partial_{i+3} M, \ i = 0, 1, 2\} \ .$$

It turns out that for any $b \in B$ the kernel and cokernel of the closure of $D^+(b)$ are finite dimensional and that the family $\{D^+(b)\}_{b \in B}$ has a well-defined index in $K^0(B)$, which equals the generalized Maslov index of (L_0, L_1, L_2) defined as follows.

The triple (L_0, L_1, L_2) induces a nondegenerate hermitian form h on L_0: Let $v = v_1 + v_2, \ w = w_1 + w_2 \in L_0(b)$ with $v_1, w_1 \in L_1(b); \ v_2, w_2 \in L_2(b)$, then

$$h_b(v, w) := \langle v_2, I_0 w_1 \rangle \ .$$

The generalized Maslov index is the element $[L_0^+] - [L_0^-] \in K^0(B)$ where L_0^+ and L_0^- are subbundles of L_0 with $L_0^+ \oplus L_0^- = L_0$ and such that h is positive on L_0^+ and negative on L_0^-.

If B is a manifold and the bundles are smooth, then by a generalization of the Atiyah-Patodi-Singer index theorem the Chern character of the index bundle can be expressed in terms of η-forms; the local term vanishes since $\mathrm{ch}(E/S) = 0$.

We outline the definition of the η-forms.

Let ∂ be the differentiation operator on $C^\infty([0,1], \mathbb{C}^{2d})$.

For $i \neq j$ and any $b \in B$ the operator $D_I(b) := I_0 \partial$ with domain

$$\mathrm{dom}\, D_I(b) := \{ s \in C^\infty([0,1], \mathbb{C}^{2d}) \mid s(0) \in L_i(b),\ s(1) \in L_j(b) \}$$

is essentially selfadjoint and its closure has a bounded inverse on $L^2([0,1], \mathbb{C}^{2d})$.

There is a family of rescaled superconnections A_t^I associated to the family of operators σD_I, where σ is a formal parameter of degree 1 with $\sigma^2 = 1$.

The η-form

$$\eta(L_i, L_j) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \in \Omega^*(B) ,$$

with $\mathrm{Tr}_\sigma(a + \sigma b) := \mathrm{Tr}(a)$, is well-defined.

The statement of the index theorem is:

$$\mathrm{ch}(\mathrm{ind}\, D^+) = [\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)] \in H_{dR}^*(B) .$$

1.3. The algebra of differential forms

In this section we study the noncommutative analogues of the algebra $\Omega^*(B)$, the Chern character and the de Rham cohomology $H_{dR}^*(B)$.

1.3.1. The universal graded differential algebra. Let $\mathcal{B}$ be an involutive locally m-convex Fréchet algebra with unit. In particular, the multiplication $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $(a,b) \mapsto ab$ is continuous, the group of invertible elements $\mathrm{Gl}(\mathcal{B})$ in $\mathcal{B}$ is open and the map $\mathrm{Gl}(\mathcal{B}) \to \mathrm{Gl}(\mathcal{B})$, $a \mapsto a^{-1}$ is continuous [**Ma**]. In this section we recall the definition of the topological universal graded differential algebra $\hat{\Omega}_* \mathcal{B}$ and collect its main properties [**Ka**] [**CQ**].

We write $\otimes_\pi$ for the completed projective tensor product.

Let

$$\hat{\Omega}_k \mathcal{B} := \mathcal{B} \otimes_\pi (\otimes_\pi^k (\mathcal{B}/\mathbb{C}))$$

and

$$\hat{\Omega}_* \mathcal{B} := \prod_{k=0}^\infty \hat{\Omega}_k \mathcal{B} .$$

With the following structures $\hat{\Omega}_* \mathcal{B}$ is an involutive Fréchet locally m-convex graded differential algebra:

Product: There is a graded continuous product on $\hat{\Omega}_* \mathcal{B}$ defined for elementary tensors by

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)(b_{k+1} \otimes b_{k+2} \otimes \ldots b_n) := \sum_{j=0}^k (-1)^{k-j} (b_0 \otimes b_1 \otimes \ldots \otimes b_j b_{j+1} \otimes b_{j+2} \otimes \ldots b_n) .$$

Differential: There is a continuous differential d of degree one on the graded algebra $\hat{\Omega}_*\mathcal{B}$ defined on elementary tensors by

$$\mathrm{d}(b_0 \otimes b_1 \otimes \ldots \otimes b_k) := 1 \otimes b_0 \otimes b_1 \otimes \ldots \otimes b_k .$$

It satisfies the graded Leibniz rule: For $\alpha \in \hat{\Omega}_k\mathcal{B}$ and $\beta \in \hat{\Omega}_*\mathcal{B}$

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\,\alpha)\beta + (-1)^k \alpha(\mathrm{d}\,\beta) .$$

Furthermore in $\hat{\Omega}_k\mathcal{B}$

$$b_0 \otimes b_1 \otimes \ldots \otimes b_k = b_0 \,\mathrm{d}\, b_1 \,\mathrm{d}\, b_2 \ldots \mathrm{d}\, b_k .$$

If $\mathcal{B}$ is a Banach algebra, then d is a map of norm one.

Involution: We extend the $*$-operation on $\mathcal{B}$ to a continuous involution on $\hat{\Omega}_*\mathcal{B}$ by setting

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)^* := (1 \otimes b_k^* \otimes b_{k-1}^* \otimes \ldots \otimes b_1^*)b_0^*$$

or equivalently

$$(b_0 \,\mathrm{d}\, b_1 \,\mathrm{d}\, b_2 \ldots \mathrm{d}\, b_k)^* = (\mathrm{d}\, b_k^* \,\mathrm{d}\, b_{k-1}^* \ldots \mathrm{d}\, b_1^*)b_0^* .$$

For $\omega_1, \omega_2 \in \hat{\Omega}_*\mathcal{B}$

$$(\omega_1\omega_2)^* = \omega_2^*\omega_1^* ,$$

and for $\omega \in \hat{\Omega}_k\mathcal{B}$

$$(\mathrm{d}\,\omega)^* = (-1)^k \,\mathrm{d}(\omega^*) .$$

Let

$$\hat{\Omega}_{\leq m}\mathcal{B} := \hat{\Omega}_*\mathcal{B} / \prod_{k=m+1}^{\infty} \hat{\Omega}_k\mathcal{B} .$$

The above structures are well-defined on $\hat{\Omega}_{\leq m}\mathcal{B}$ as well.

We identify $\hat{\Omega}_{\leq m}\mathcal{B}$ as a graded vector space with the subspace

$$\{\omega \in \hat{\Omega}_*\mathcal{B} \mid \omega^n = 0 \text{ for } n > m\} \subset \hat{\Omega}_*\mathcal{B} ,$$

where ω^n denotes the homogeneous part of degree n of $\omega \in \hat{\Omega}_*\mathcal{B}$.

The Fréchet space $\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$, generated by the supercommutators in $\hat{\Omega}_*\mathcal{B}$, is preserved by d by Leibniz rule. It follows that $(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d})$ is a complex.

DEFINITION 1.3.1. *The* DE RHAM HOMOLOGY *of* $\mathcal{B}$ *is*

$$H_*^{dR}(\mathcal{B}) := H_*(\hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}, \mathrm{d}) .$$

On the right hand side we take the topological homology, i.e. we quotient out the closure of the range of d *in order to obtain a Hausdorff space.*

The map d induces maps $\mathrm{d} : (\hat{\Omega}_*\mathcal{B})^n \to (\hat{\Omega}_*\mathcal{B})^n$ and $\mathrm{d} : M_n(\hat{\Omega}_*\mathcal{B}) \to M_n(\hat{\Omega}_*\mathcal{B})$, $A \mapsto \mathrm{d}(A)$ by applying d componentwise. Note the difference between $\mathrm{d}\, A = \mathrm{d} \circ A$ and $\mathrm{d}(A)$. Sometimes we write $(\mathrm{d}\, A)$ for $\mathrm{d}(A)$.

For $A \in M_n(\hat{\Omega}_k\mathcal{B})$

$$\mathrm{d}\, A = (\mathrm{d}\, A) + (-1)^k A\,\mathrm{d} .$$

The trace

$$\mathrm{tr} : M_n(\hat{\Omega}_*\mathcal{B}) \to \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

is defined by adding up the diagonal elements. It vanishes on supercommutators.

In the following definition and proposition a projection is not assumed to be selfadjoint.

DEFINITION 1.3.2. *Let $P \in M_n(\mathcal{B})$ be a projection. Then*

$$\mathrm{ch}(P) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \operatorname{tr} P(\mathrm{d}(P))^{2k} \in \hat{\Omega}_* \mathcal{B} / \overline{[\hat{\Omega}_* \mathcal{B}, \hat{\Omega}_* \mathcal{B}]_s}$$

is the CHERN CHARACTER FORM *of P.*

PROPOSITION 1.3.3.   (1) *The Chern character form is closed.*
  (2) *If $P : [0,1] \to M_n(\mathcal{B})$ is a differentiable path of projections, then $\mathrm{ch}(P_1) - \mathrm{ch}(P_0)$ is exact.*
  (3) *If $\mathcal{B}$ is a local Banach algebra* [**Bl**]*, then the Chern character form induces a homomorphism*

$$\mathrm{ch} : K_0(\mathcal{B}) \to H_*^{dR}(\mathcal{B}), \ \mathrm{ch}([P] - [Q]) := \mathrm{ch}(P) - \mathrm{ch}(Q) \ ,$$

  called the CHERN CHARACTER.

PROOF. First note that for a projection $P \in M_n(\mathcal{B})$

$$0 = \mathrm{d}((1 - P)P) = (1 - P)(\mathrm{d}\,P) - (\mathrm{d}\,P)P \ ,$$

hence $(1 - P)(\mathrm{d}\,P) = (\mathrm{d}\,P)P$ and $P(\mathrm{d}\,P)P = 0$. Therefore

$$P(\mathrm{d}\,P)^2 = (\mathrm{d}\,P)^2 P \ .$$

Analogous formulas hold for the derivative of a path of projections.
  (1) follows from

$$\begin{aligned}
\mathrm{d}\operatorname{tr} P(\mathrm{d}\,P)^{2k} &= \operatorname{tr}(\mathrm{d}\,P)^{2k+1} \\
&= \operatorname{tr}(1 - P)(\mathrm{d}\,P)^{2k+1}(1 - P) + \operatorname{tr} P(\mathrm{d}\,P)^{2k+1} P \\
&= 0 \ .
\end{aligned}$$

  (2) We have that

$$\begin{aligned}
(\operatorname{tr} P(\mathrm{d}\,P)^{2k})' &= \operatorname{tr} P'(\mathrm{d}\,P)^{2k} + \operatorname{tr} P((\mathrm{d}\,P)^{2k})' \\
&= \sum_{i=0}^{2k-1} \operatorname{tr} P(\mathrm{d}\,P)^i (\mathrm{d}\,P')(\mathrm{d}\,P)^{2k-i-1} \ .
\end{aligned}$$

For i even

$$\begin{aligned}
&\operatorname{tr} P(\mathrm{d}\,P)^i (\mathrm{d}\,P')(\mathrm{d}\,P)^{2k-i-1} \\
&= \operatorname{tr}(\mathrm{d}\,P)^i (\mathrm{d}(PP'))(\mathrm{d}\,P)^{2k-i-1} - \operatorname{tr}(\mathrm{d}\,P)^i (\mathrm{d}\,P)P'(\mathrm{d}\,P)^{2k-i-1} \\
&= \operatorname{tr}(\mathrm{d}\,P)^i (\mathrm{d}(PP'))(\mathrm{d}\,P)^{2k-i-1} \\
&= \mathrm{d}\operatorname{tr} P(\mathrm{d}\,P)^{i-1}(\mathrm{d}(PP'))(\mathrm{d}\,P)^{2k-i-1} \ .
\end{aligned}$$

For i odd the argument is similar.
  (3) follows from (1) and (2).   □

1.3.2. Supercalculus. Let $\mathcal{B}$ be as before.

In this section all spaces and tensor products are $\mathbb{Z}/2$-graded. If no grading is specified we assume the grading to be trivial.

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2$-graded complex vector space with $\dim V^+ = m$, $\dim V^- = n$ and consider $\hat{\Omega}_*\mathcal{B}$ as a $\mathbb{Z}/2$-graded space with the grading induced by the form degree.

The space $V \otimes \hat{\Omega}_*\mathcal{B}$ is a free $\mathbb{Z}/2$-graded right $\hat{\Omega}_*\mathcal{B}$-module. It is furthermore a left supermodule of the superalgebra $\operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$. (Note that our setting differs from the corresponding one in [**BGV**] since we deal with right $\hat{\Omega}_*\mathcal{B}$-modules. This leads to different signs.)

The supertrace $\operatorname{tr}_s : \operatorname{End}(V) \to \mathbb{C}$ extends to a supertrace

$$\operatorname{tr}_s : \operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B} \to \hat{\Omega}_*\mathcal{B}/\overline{[\hat{\Omega}_*\mathcal{B}, \hat{\Omega}_*\mathcal{B}]_s}$$

$$\operatorname{tr}_s(T \otimes \omega) := \operatorname{tr}_s(T)\omega \ .$$

By quotienting out the supercommutator we ensure that $\operatorname{tr}_s([T_1, T_2]_s) = 0$ for $T_1, T_2 \in \operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$.

The differential d acts on elements of $V \otimes \hat{\Omega}_*\mathcal{B}$ resp. $\operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$ by

$$\mathrm{d}(A \otimes \omega) = (-1)^{\deg A} A \otimes \mathrm{d}\,\omega$$

for $A \in V^{\pm}$ resp. $A \in \operatorname{End}^{\pm}(V)$ and $\omega \in \hat{\Omega}_*\mathcal{B}$.

Though the differential d is not a right $\hat{\Omega}_*\mathcal{B}$-module map, the supercommutator $[\mathrm{d}, T]_s$ with $T \in \operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$ is in $\operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$; namely

$$[\mathrm{d}, T]_s = \mathrm{d}(T) \ .$$

Since $\operatorname{tr}_s(A \otimes \omega) = 0$ for $A \in \operatorname{End}^-(V)$, we have that

$$\operatorname{tr}_s[\mathrm{d}, T]_s = \operatorname{tr}_s \mathrm{d}(T) = \mathrm{d}\operatorname{tr}_s T$$

for $T \in \operatorname{End}(V) \otimes \hat{\Omega}_*\mathcal{B}$.

See §5.2.3, §5.2.4, §5.2.5 for notions of Hilbert-Schmidt operators, trace class operators and adjointable operators used in the following.

If M is a complete Riemannian manifold and T is a trace class operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{B})$, then

$$\operatorname{Tr}_s T := \int_M \operatorname{tr}_s \overline{R}(T)(x) \; dx \in \hat{\Omega}_{\leq\mu}\mathcal{B}/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{B}, \hat{\Omega}_{\leq\mu}\mathcal{B}]_s} \ .$$

Note that

$$\operatorname{Tr}_s[A, B]_s = 0$$

if A is an adjointable operator and B a trace class operator or if A, B are Hilbert-Schmidt operators.

1.3.3. The algebras $\mathcal{A}_\infty$ and $\hat{\Omega}_*\mathcal{A}_\infty$. Let $(\mathcal{A}_j, \iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j)_{j \in \mathbb{N}_0}$ be a projective system of involutive Banach algebras with unit satisfying the following conditions (see [**Lo3**], §2.1):

- The algebra $\mathcal{A} := \mathcal{A}_0$ is a C^*-algebra.
- For any $j \in \mathbb{N}_0$ the map $\iota_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j$ is injective.
- For any $j \in \mathbb{N}_0$ the map $\iota_j : \mathcal{A}_\infty := \varprojlim_i \mathcal{A}_i \to \mathcal{A}_j$ has dense range.

- For any $j \in \mathbb{N}_0$ the algebra $\mathcal{A}_j$ is stable with respect to the holomorphic functional calculus in $\mathcal{A}$.

The projective limit $\mathcal{A}_\infty$ is an involutive locally m-convex Fréchet algebra with unit.

The motivating example is $\mathcal{A}_j = C^j(B)$ for a closed smooth manifold B.

PROPOSITION 1.3.4. *For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we have:*

(1) *The map $\iota_j : \mathcal{A}_\infty \to \mathcal{A}_j$ is injective.*

(2) *The algebras $M_n(\mathcal{A}_\infty)$ and $M_n(\mathcal{A}_j)$ are stable with respect to the holomorphic functional calculus in $M_n(\mathcal{A})$.*

(3) *The map $\iota_{0*} : K_0(\mathcal{A}_\infty) \to K_0(\mathcal{A})$ is an isomorphism.*

PROOF. (1) follows immediately.

(2) follows from [**Bo**], Prop. A.2.2.

(3) follows from [**Bo**], Th. A.2.1. $\square$

The projective system $(\mathcal{A}_j, \iota_{j+1,j})_{j \in \mathbb{N}_0}$ induces two projective systems of involutive graded differential Fréchet algebras.

One of them is given by the maps

$$\iota_{j+1,j*} : \hat{\Omega}_* \mathcal{A}_{j+1} \to \hat{\Omega}_* \mathcal{A}_j .$$

Furthermore $(\hat{\Omega}_{\leq m} \mathcal{A}_j)_{m,j \in \mathbb{N}}$ is a projective system of involutive Banach graded differential algebras.

Their limits coincide:

The inclusion $\hat{\Omega}_{\leq m} \mathcal{A}_j \to \hat{\Omega}_* \mathcal{A}_j$ is left inverse to the projection $\hat{\Omega}_* \mathcal{A}_j \to \hat{\Omega}_{\leq m} \mathcal{A}_j$. The induced maps between the projective limits are inverse to each other. It follows that

$$\varprojlim_{j} \hat{\Omega}_* \mathcal{A}_j \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m} \mathcal{A}_j .$$

Furthermore the inclusions $\iota_{j*} : \hat{\Omega}_* \mathcal{A}_\infty \to \hat{\Omega}_* \mathcal{A}_j$ induce a map

$$\iota_* : \hat{\Omega}_* \mathcal{A}_\infty \to \varprojlim_{j} \hat{\Omega}_* \mathcal{A}_j .$$

PROPOSITION 1.3.5. *There are the following canonical isomorphisms of involutive Fréchet locally m-convex graded differential algebras:*

(1) $\hat{\Omega}_* \mathcal{A}_\infty \cong \varprojlim_{j} \hat{\Omega}_* \mathcal{A}_j \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m} \mathcal{A}_j .$

There are the following canonical isomorphisms of graded Fréchet spaces:

(2) $\hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s} \cong \varprojlim_{j,m} \hat{\Omega}_{\leq m} \mathcal{A}_j / \overline{[\hat{\Omega}_{\leq m} \mathcal{A}_j, \hat{\Omega}_{\leq m} \mathcal{A}_j]_s} ,$

(3) $H_*^{dR}(\mathcal{A}_\infty) \cong \varprojlim_{j} H_*^{dR}(\mathcal{A}_j) \cong \varprojlim_{j,m} H_*(\hat{\Omega}_{\leq m} \mathcal{A}_j / \overline{[\hat{\Omega}_{\leq m} \mathcal{A}_j, \hat{\Omega}_{\leq m} \mathcal{A}_j]_s}, d) .$

PROOF. (1) It is enough to prove that the right hand side and the left hand side are isomorphic as topological vector spaces. This follows from the fact that projective limits and projective tensor products commute ([**Kö**], 41.6).

(2) and (3) follow from the three technical lemmas below. $\square$

The importance of this proposition for our purposes is the following: Since the analysis is easier on Banach spaces than on Fréchet spaces we will prove the index

theorem in $H_*(\hat{\Omega}_{\leq m}\mathcal{A}_j/\overline{[\hat{\Omega}_{\leq m}\mathcal{A}_j, \hat{\Omega}_{\leq m}\mathcal{A}_j]_s}, \mathrm{d})$, making sure that the expressions in the index theorem behave well under the projective limit. By the proposition this will prove the index theorem in $H_*^{dR}(\mathcal{A}_\infty)$.

LEMMA 1.3.6. Let $(V_n, f_{n+1,n} : V_{n+1} \to V_n)_{n \in \mathbb{N}}$ be a projective system of Banach spaces with projective limit V_∞. Let $(A_n, f_{n+1,n}|_{A_{n+1}})_{n \in \mathbb{N}}$ be a projective subsystem of sets such that for any $n \in \mathbb{N}$ the range of the induced map $f_n : A_\infty := \varprojlim_n A_n \to A_n$ is dense in A_n. Then

$$\varprojlim_n \overline{A_n} = \overline{\varprojlim_n A_n} \ .$$

Hence a subset S_1 of V_∞ is dense in $S_2 \subset V_\infty$ if and only if $f_n S_1 \subset V_n$ is dense in $f_n S_2 \subset V_n$ for any $n \in \mathbb{N}$.

PROOF. Without loss of generality we may assume that the maps $f_{n+1,n}$ have norm less than or equal to one. Let $a \in \varprojlim_n \overline{A_n}$. Then for any $n \in \mathbb{N}$ there is $b_n \in A_\infty$ with $|f_n(b_n) - f_n(a)| \leq \frac{1}{n}$. The sequence $(b_n)_{n \in \mathbb{N}}$ converges in V_∞ to a. Hence $\overline{A_\infty} = \varprojlim_n \overline{A_n}$.   □

LEMMA 1.3.7.   (1) Let $(V_n, f_{n+1,n})_{n \in \mathbb{N}}$, $(W_n, g_{n+1,n})_{n \in \mathbb{N}}$ be projective systems of Banach spaces. Let $(d_n : V_n \to W_n)_{n \in \mathbb{N}}$ be a morphism of projective systems with induced map $d_\infty : V_\infty \to W_\infty$. Then

$$\mathrm{Ker}\, d_\infty = \varprojlim_n \mathrm{Ker}\, d_n \ .$$

(2) In the situation of (1) assume furthermore that the induced maps $f_n : V_\infty \to V_n$ and $g_n : W_\infty \to W_n$ have dense range for all $n \in \mathbb{N}$. Then

$$\overline{\mathrm{Ran}\, d_\infty} = \varprojlim_n \overline{\mathrm{Ran}\, d_n} \ .$$

(3) Let $(A_n, f_{n+1,n})_{n \in \mathbb{N}}$ be a projective system of Banach algebras such that for any $n \in \mathbb{N}$ the range of $f_n : A_\infty := \varprojlim_n A_n \to A_n$ is dense in A_n.

Then

$$\overline{[A_\infty, A_\infty]} = \varprojlim_n \overline{[A_n, A_n]} \ .$$

PROOF. (1) Let $x \in \mathrm{Ker}\, d_\infty$. Then $d_n f_n(x) = g_n d_\infty(x) = 0$.
Conversely if $d_n f_n x = 0$ for all $n \in \mathbb{N}$, then by definition $d_\infty x = 0$.
(2) follows from the previous lemma since $g_n \mathrm{Ran}\, d_\infty$ is dense in $\overline{\mathrm{Ran}\, d_n}$.
(3) follows from the previous lemma since $f_n[A_\infty, A_\infty]$ is dense in $\overline{[A_n, A_n]}$ for all $n \in \mathbb{N}$.   □

LEMMA 1.3.8. Let $(V_n, f_{n+1,n})_{n \in \mathbb{N}}$ be a projective system of Banach spaces and let $(A_n, f_{n+1,n}|_{A_{n+1}})_{n \in \mathbb{N}}$ be a projective subsystem such that A_n is a closed subspace of V_n for all $n \in \mathbb{N}$.
Let $V_\infty := \varprojlim_n V_n$ and $A_\infty := \varprojlim_n A_n$ and assume furthermore that the image of A_{n+1} is dense in A_n for any $n \in \mathbb{N}$.
Then, canonically,

$$V_\infty/A_\infty \cong \varprojlim_n V_n/A_n \ .$$

PROOF. For $n \in \mathbb{N}$ let $f_n : V_\infty \to V_n$ be the induced map. We prove that the map

$$f_\infty : V_\infty / A_\infty \to \varprojlim_n V_n / A_n$$

is an isomorphism:

For injectivity let $[v] \in V_\infty / A_\infty$ with $f_\infty[v] = 0$. Then $f_n v \in A_n$ for all $n \in \mathbb{N}$, so $v \in A_\infty$, thus $[v] = 0$.

In order to prove surjectivity we assume that the norms of the maps $f_{n+1,n}$ are less than or equal to one, which can be obtained by rescaling the norms inductively.

Let $v \in \varprojlim_n V_n / A_n$. Choose inductively $v_n \in V_n$ such that $[v_n] \in V_n / A_n$ is the image of v with respect to the map $\varprojlim_n V_n / A_n \to V_n / A_n$ and such that

$$|f_{n+1,n} v_{n+1} - v_n| \le n^{-1} .$$

(Here we use that $f_{n+1,n} A_{n+1}$ is dense in A_n.)

For $n \ge j$ let $f_{n,j} : V_n \to V_j$ be the induced map. For any $j \in \mathbb{N}$ the sequence $(f_{n,j}(v_n))_{n \ge j}$ converges to an element $\tilde{v}_j \in V_j$. We have that $f_{j+1,j} \tilde{v}_{j+1} = \tilde{v}_j$, hence there is an element $\tilde{v} \in V_\infty$ with $f_j \tilde{v} = \tilde{v}_j$. Since $[\tilde{v}_j] = [v_j] \in V_j / A_j$, we have that $f_\infty[\tilde{v}] = v$.

$\square$

1.4. Lagrangian projections

Let $\mathcal{A}$ be a unital C^*-algebra.

In this section we define and study the analogues of Lagrangian subbundles and of the Maslov index bundle.

Let $n \in \mathbb{N}$.

DEFINITION 1.4.1. *Two selfadjoint projections $P_1, P_2 \in M_n(\mathcal{A})$ are called* TRANSVERSE *if*

$$\operatorname{Ran} P_1 \oplus \operatorname{Ran} P_2 = \mathcal{A}^n .$$

We will often use the following transversality criterion: Two selfadjoint projections P_1, P_2 are transverse if and only there exist $a, b \in \operatorname{Gl}(\mathcal{A})$ such that $aP_1 + bP_2 \in M_n(\mathcal{A})$ is invertible. This is equivalent to the invertibility of $aP_1 + bP_2$ for any $a, b \in \operatorname{Gl}(\mathcal{A})$.

If $\mathcal{A} = C(B)$ for a compact space B, then the transversality of two projections is equivalent to the transversality of the corresponding subbundles.

1.4.1. Definition and properties. Let $\mathcal{A}^{2n}$ be endowed with the standard $\mathcal{A}$-valued scalar product and let

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : \mathcal{A}^n \oplus \mathcal{A}^n \to \mathcal{A}^n \oplus \mathcal{A}^n .$$

DEFINITION 1.4.2. *A* LAGRANGIAN PROJECTION ON $\mathcal{A}^{2n}$ *is a selfadjoint projection $P \in M_{2n}(\mathcal{A})$ with*

$$PI_0 = I_0(1 - P) .$$

If $\mathcal{A} = C(B)$ for some compact space B, then the Lagrangian projections are in one-to-one correspondence with the subbundles of $B \times \mathbb{C}^{2n}$ that are Lagrangian with respect to the skew-hermitian form induced by I_0.

Note furthermore that any Lagrangian projection P on $\mathcal{A}^{2n}$ is also a Lagrangian projection on the Hilbert $M_n(\mathcal{A})$-module $M_n(\mathcal{A})^2$.

Denote

$$P_s := \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

LEMMA 1.4.3. (1) *For every Lagrangian projection P on $\mathcal{A}^{2n}$ there is a unitary $p \in M_n(\mathcal{A})$ such that*

$$P = \tfrac{1}{2} \begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} .$$

(2) *For every Lagrangian projection P on $\mathcal{A}^{2n}$ the unitary*

$$U = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix} \in M_{2n}(\mathcal{A})$$

with p as in (1) fulfills $U I_0 = I_0 U$ and $U P U^* = P_s$.

PROOF. (1) Since P is selfadjoint, there are $a, b, c \in M_n(\mathcal{A})$ with $a = a^*, c = c^*$ such that $P = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. From $P I_0 = I_0(1 - P)$ it follows that

$$\begin{pmatrix} ia & -ib \\ ib^* & -ic \end{pmatrix} = \begin{pmatrix} i(1 - a) & -ib \\ ib^* & -i(1 - c) \end{pmatrix} ,$$

thus $a = c = \tfrac{1}{2}$. Furthermore from $P^2 = P$ it follows that $2b$ is unitary.

(2) is clear. □

Let $\mathcal{A}_\infty$ be as in the previous section.

LEMMA 1.4.4. *Let $P \in M_{2n}(\mathcal{A}_\infty)$ be a Lagrangian projection of $\mathcal{A}^{2n}$ transverse to P_s. Let $\tilde{P} \in M_{2n}(\mathbb{C})$ be a complex Lagrangian projection. Then for every $0 < \varepsilon_1 < \varepsilon_2$ there is a smooth path of unitaries $U : [0, \varepsilon_2] \to M_{2n}(\mathcal{A}_\infty)$ such that*

(1) $U(0) P U(0)^* = \tilde{P}$,
(2) U *equals 1 on a neighborhood of ε_2,*
(3) U *is constant on $[0, \varepsilon_1]$,*
(4) U *is diagonal with respect to the decomposition $\mathcal{A}^{2n} = \mathcal{A}^n \oplus \mathcal{A}^n$.*

Note that (4) implies that $U I_0 = I_0 U$.

PROOF. It is enough to prove the assertion for $\tilde{P} = P_s$.

For P let p be as in the previous lemma.

Since P and P_s are transverse, $P - P_s$ is invertible. It follows that $p - 1$ and $p^* - 1$ are invertible, so $\log p$ and $\log p^*$ are well-defined if we choose the complement of $[0, \infty)$ in $\mathbb{C}$ as a domain for the logarithm.

Let $\chi : [0, \varepsilon_2] \to [0, 1]$ be a smooth function with $\chi|_{[0,\varepsilon_1]} = 0$ and $\chi(t) = 1$ for $t \in [\frac{\varepsilon_2 - \varepsilon_1}{2}, \varepsilon_2]$. The smooth path of unitaries

$$\gamma : [0, \varepsilon_2] \to M_n(\mathcal{A}), \ \gamma(t) := \exp(2\pi i \chi(t) + (1 - \chi(t)) \log p^*)$$

connects p^* with 1. Moreover $\gamma(t) \in M_n(\mathcal{A}_\infty)$ for all $t \in [0, \varepsilon_2]$ since $M_n(\mathcal{A}_\infty)$ is stable under holomorphic function calculus in $M_n(\mathcal{A})$ by Prop. 1.3.4.

Then

$$U := \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

satisfies the conditions. □

1.4.2. The Maslov index. Let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections on $\mathcal{A}^{2n}$. For $x \in \mathcal{A}^{2n}$ write $x = x_1 + x_2$ with $x_i \in \operatorname{Ran} P_i$, $i = 1, 2$.

The form
$$h : \operatorname{Ran} P_0 \times \operatorname{Ran} P_0 \to \mathcal{A}, \ (x, y) \mapsto \langle x_2, I_0 y_1 \rangle$$
is hermitian and its radical vanishes [**Wa**]. Since $x_1 = P_1(P_1 + P_2)^{-1} x$ and $x_2 = P_2(P_1 + P_2)^{-1} x$, the corresponding selfadjoint matrix is
$$A := P_0(P_1 + P_2)^{-1} P_2 I_0 P_1 (P_1 + P_2)^{-1} P_0 \in M_{2n}(\mathcal{A}) .$$

Since
$$
\begin{aligned}
A &= P_0(P_1 + P_2)^{-1} P_2 I_0 P_1 (P_1 + P_2)^{-1} P_0 \\
&= P_0(P_1 + P_2)^{-1}(P_1 + P_2) I P_1 (P_1 + P_2)^{-1}(P_0 + P_2) \\
&= (P_0 + P_1) I P_1 (P_1 + P_2)^{-1}(P_0 + P_1) ,
\end{aligned}
$$
the range of A is closed. Hence the hermitian form h is non-singular and defines an element in $K_0(\mathcal{A})$ [**Ros2**].

DEFINITION 1.4.5. *The* MASLOV INDEX $\tau(P_0, P_1, P_2) \in K_0(\mathcal{A})$ *of a triple of pairwise transverse Lagrangian projections* (P_0, P_1, P_2) *is the class of the hermitian form* h *in* $K_0(\mathcal{A})$.

We can express the Maslov index in terms of A as follows:
$$\tau(P_0, P_1, P_2) = [1_{\{x>0\}}(A)] - [1_{\{x<0\}}(A)] \in K_0(\mathcal{A}) .$$
Note that the Maslov index is invariant under even permutations and changes sign under odd permutations.

PROPOSITION 1.4.6. *Let* $P_i : [0, 1] \to M_{2n}(\mathcal{A})$, $i = 0, 1, 2$, *be continuous paths of Lagrangian projections such that* $P_i(t) - P_j(t)$ *is invertible for* $i \neq j$ *and all* $t \in [0, 1]$.
Then the Maslov index $\tau(P_0(t), P_1(t), P_2(t))$ *does not depend on* t.

PROOF. The selfadjoint element $A(t) \in M_{2n}(\mathcal{A})$ defined by $(P_0(t), P_1(t), P_2(t))$ as above depends continuously on t for all $t \in [0, 1]$. It follows that the projections $1_{\{x>0\}}(A(t))$ and $1_{\{x<0\}}(A(t))$ also depend continuously on t, thus their K-theory classes are constant. $\qquad\square$

Let B be a compact space and let (P_0, P_1, P_2) be a triple of pairwise transverse Lagrangian projections in $M_{2n}(C(B))$. Let (L_0, L_1, L_2) be the corresponding triple of Lagrangian subbundles of $B \times \mathbb{C}^{2n}$. Then the Maslov index bundle $[L_0^+] - [L_0^-]$ defined in §1.2 corresponds to $\tau(P_0, P_1, P_2)$ under the canonical isomorphism $K^0(B) \cong K_0(C(B))$.

Now we study in some detail the Maslov index of a triple (P_0, P_1, P_2) with $P_0 = P_s$. The general case can be reduced to this case by Lemma 1.4.3.

The Cayley transform $a \mapsto \frac{a-i}{a+i}$, defined for selfadjoint $a \in M_n(\mathcal{A})$, yields a bijective map
$$a \mapsto P(a) := \tfrac{1}{2} \begin{pmatrix} 1 & \frac{a+i}{a-i} \\ \frac{a-i}{a+i} & 1 \end{pmatrix} .$$
from the space of selfadjoint elements in $M_n(\mathcal{A})$ to space of projections in $M_{2n}(\mathcal{A})$ transverse to P_s.

LEMMA 1.4.7. *Let $a_1, a_2 \in M_n(\mathcal{A})$ be selfadjoint. Then $P(a_1)$ and $P(a_2)$ are transverse projections if and only if $a_1 - a_2$ is invertible.*

PROOF. Let $U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

$$UP(a_j)U^* = (a_j^2 + 1)^{-1} \begin{pmatrix} a_j^2 & -ia_j \\ ia_j & 1 \end{pmatrix} .$$

Now $P(a_1)$ and $P(a_2)$ are transverse if and only if

$$(a_1^2 + 1)UP(a_1)U^* - (a_2^2 + 1)UP(a_2)U^*$$

is invertible and this is the case if and only if $a_1 - a_2$ is invertible.   □

LEMMA 1.4.8. *Let (P_s, P_1, P_2) be a triple of pairwise transverse Lagrangian projections and let $a_1, a_2 \in M_n(\mathcal{A})$ be such that $P_i = P(a_i)$, $i = 1, 2$. Let $p^+ := 1_{\{x>0\}}(a_1 - a_2)$.*

   (1) *There are continuous paths $P_1, P_2 : [0, 2] \to M_{2n}(\mathcal{A})$ of Lagrangian projections such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0, 2]$ and such that $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$.*
   (2)
$$\tau(P_s, P_1, P_2) = [p^+] - [1 - p^+] .$$

PROOF. (1) For $t \in [0, 1]$ define $a_1(t) := t(a_1 - a_2) + (1 - t)a_1$ and $a_2(t) := t(a_2 - a_1) + (1 - t)a_2$. Then $a_1(t) - a_2(t) = (1 + t)(a_1 - a_2)$ is invertible, thus the projections $P(a_1(t))$ and $P(a_2(t))$ are transverse. Furthermore $a_1(0) = a_1$ and $a_1(1) = a_1 - a_2$, whereas $a_2(0) = a_2$ and $a_2(1) = a_2 - a_1$.

For $t \in [1, 2]$ let $a_1(t)$ be a path of invertible selfadjoint elements with $a_1(1) = a_1 - a_2$ and $a_1(2) = p^+ - (1 - p^+)$ and let $a_2(t) = -a_1(t)$.

Then the paths $P(a_1(t))$ and $P(a_2(t))$ satisfy the conditions.

(2) From (1) and Prop. 1.4.6 it follows that $\tau(P_s, P_1, P_2) = \tau(P_s, P(2p^+ - 1), P(1 - 2p^+))$.

The Cayley transform of $2p^+ - 1$ is $i(2p^+ - 1)$. By computing the matrix A we see that $1_{\{x>0\}}(A) = p^+$ and $1_{\{x<0\}}(A) = (1 - p^+)$.   □

The Fredholm Operator and Its Index

2.1. The operator D on M

2.1.1. Definition of D. Now we come back to the geometric situation described in §1.1.

By taking the tensor product of the bundle E with the C^*-algebra $\mathcal{A}$ we obtain an $\mathcal{A}$-vector bundle [**MF**]. Furthermore we consider the bundle $E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$, $i, \mu \in \mathbb{N}_0$, of right $\hat{\Omega}_{\leq \mu} \mathcal{A}_i$-modules. Keep in mind that E can be trivialized on M via a global orthonormal frame. Thus no theory of Banach space bundles is needed in this context.

The hermitian metric on E extends to an $\mathcal{A}$-valued scalar product $\langle \cdot, \cdot \rangle$ on $E \otimes \mathcal{A}$ and to an $\hat{\Omega}_{\leq \mu} \mathcal{A}_i$-valued non-degenerated product on $E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$ (see §5.2.3 for this notion). Furthermore parallel transport is defined on $E \otimes \mathcal{A}$ resp. $E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$.

By the trivialization of $E|_F$ fixed in §1.1 we identify $(E|_F \otimes \mathcal{A}, I)_x$, $x \in F$, with $(\mathcal{A}^{4d}, I_0 \oplus (-I_0))$ as a Hilbert $\mathcal{A}$-module with a skew-hermitian structure, and $(E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)_x$, $x \in F$, with $(\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}$ as a right $\hat{\Omega}_{\leq \mu} \mathcal{A}_i$-module with an $\hat{\Omega}_{\leq \mu} \mathcal{A}_i$-valued non-degenerated product.

Recall that the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$ can be defined as the completion of $C_c^\infty(M, E \otimes \mathcal{A})$ with respect to the norm induced by the $\mathcal{A}$-valued scalar product

$$\langle f, g \rangle := \int_M \langle f(x), g(x) \rangle \, dx \ .$$

(For fixing notation a short introduction to Hilbert $\mathcal{A}$-modules can be found in §5.1.1.) By Prop. 5.1.18 any orthonormal basis of the Hilbert space $L^2(M, E)$ is an orthonormal basis of $L^2(M, E \otimes \mathcal{A})$ and $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$.

We introduce the Schwartz space of sections of $E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$:
For $k \in \mathbb{Z}/6$ define the Schwartz space

$$\mathcal{S}(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) := \mathcal{S}(\mathbb{R}) \otimes_\pi C^\infty([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) \ .$$

For $r \geq 0$, $0 \leq b \leq \frac{1}{4}$ choose a partition of unity $\{\phi_k\}_{k \in J}$ subordinate to the covering $\mathcal{U}(r, b)$.

The embedding $F_k(r, b) \hookrightarrow Z_k$, $k \in \mathbb{Z}/6$, and the trivialization of $E|_F$ induce a map

$$C^\infty(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) \to C^\infty(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}), \ f \mapsto \phi_k f \ .$$

As a vector space let $\mathcal{S}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ be the largest subspace of $C^\infty(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ such that for all $k \in \mathbb{Z}/6$ the maps

$$\mathcal{S}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) \to \mathcal{S}(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}), \ f \mapsto \phi_k f$$

are well-defined. Endowed with the seminorms of $C^\infty(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and those of $\mathcal{S}(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$ applied to $\phi_k f$ for $k \in \mathbb{Z}/6$ the space $\mathcal{S}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is a Fréchet space. The topology does not depend on the choice of r and b, neither on the choice of the partition of unity.

To a triple $R = (P_0, P_1, P_2)$ of Lagrangian projections of $\mathcal{A}^{2d}$ with $P_i \in M_{2d}(\mathcal{A}_\infty)$, $i = 0, 1, 2$, we associate boundary conditions on sections of $E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$ in the following way:

For $k \in \mathbb{N}_0 \cup \{\infty\}$ let

$$C_R^k(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) \quad := \quad \{f \in C^k(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) \mid (P_i \oplus P_i) \bar{\partial}_E^l f(x) = \bar{\partial}_E^l f(x)$$
$$\text{for } x \in \partial_i M \cup \partial_{i+3} M, \ i = 0, 1, 2; \ l \in \mathbb{N}_0, l \leq k\}$$

endowed with the subspace topology.

Analogously we define $C_{R0}^k(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and $C_{Rc}^k(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Furthermore let $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ be the vector space $\mathcal{S}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) \cap C_R^\infty(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ with the subspace topology of $\mathcal{S}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Now we introduce the operator D:

Fix a triple of pairwise transverse Lagrangian projections $R = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ of $\mathcal{A}^{2d}$ with $\mathcal{P}_i \in M_{2d}(\mathcal{A}_\infty)$, $i = 0, 1, 2$. We define D on $L^2(M, E \otimes \mathcal{A})$ as the closure of the Dirac operator $\bar{\partial}_E$ with domain $C_{Rc}^\infty(M, E \otimes \mathcal{A})$ and D^+ resp. D^- as the restriction of D to the sections of $E^+ \otimes \mathcal{A}$ resp. $E^- \otimes \mathcal{A}$.

Note that D is symmetric.

2.1.2. Comparison with D_s. Recall that $P_s = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_{2d}(\mathbb{C})$.

Fix a triple of pairwise transverse Lagrangian projections $(\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$ with $\mathcal{P}_0^s = P_s$ and $\mathcal{P}_1^s, \mathcal{P}_2^s \in M_{2d}(\mathbb{C})$. Let D_s on $L^2(M, E \otimes \mathcal{A})$ be the closure of the Dirac operator $\bar{\partial}_E$ with domain $C_{Rc}^\infty(M, E \otimes \mathcal{A})$ for the triple $R = (\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$.

Let $W \in C^\infty(M, \text{End}^+ E \otimes \mathcal{A}_\infty)$ be such that

- $WW^* = 1$,
- $W(x)(\mathcal{P}_i \oplus \mathcal{P}_i)W(x)^* = (\mathcal{P}_i^s \oplus \mathcal{P}_i^s)$ for all $x \in \partial_i M \cup \partial_{i+3} M$, $i = 0, 1, 2$,
- W is parallel on $M \setminus F$ and on a neighborhood of ∂M,
- for all $k \in \mathbb{Z}/6$ the restriction of W to $F_k(0, \frac{1}{4})$ depends only on the coordinate x_2^k,
- W commutes with the Clifford multiplication.

Some of these properties are not needed in this section, but are important for the proof of the index theorem.

PROPOSITION 2.1.1.   (1) *We have that $WDW^* = D_s + Wc(dW^*)$ with $c(dW^*)|_F := c(e_2)\partial_{e_2} W^*$ and $c(dW^*)|_{M \setminus F} := 0$. In particular, $Wc(dW^*) \in C^\infty(M, \text{End}^- E \otimes \mathcal{A}_\infty)$.*
   (2) *The operator D is regular and selfadjoint.*

PROOF. (1) For $R = (\mathcal{P}_0^s, \mathcal{P}_1^s, \mathcal{P}_2^s)$ and $f \in C_{cR}^\infty(M, E \otimes \mathcal{A})$ we have that $(WDW^* f)|_{M \setminus F} = (D_s f)|_{M \setminus F}$ and

$$\begin{aligned} (WDW^* f)|_F &= (D_s f)|_F + W[c(e_2)\partial_{e_2}, W^*]_s(f|_F) \\ &= (D_s f)|_F + Wc(e_2)(\partial_{e_2} W^*)(f|_F) \ . \end{aligned}$$

(2) The restriction of D_s to the Hilbert space $L^2(M, E)$ is selfadjoint [**BK**]. Hence $(1+D_s^2)$ has a bounded inverse on $L^2(M, E)$. It follows that the range of $(1+D_s^2)$ on $L^2(M, E \otimes \mathcal{A})$ is dense, thus D_s is regular. By an analogous argument the operators $D_s \pm i$ have dense range. From Lemma 5.1.15 it follows that D_s is selfadjoint.

By (1) the operator D is a bounded perturbation of D_s, hence D is selfadjoint and regular (see Prop. 5.1.11). $\qquad \square$

The existence of a section W fulfilling the properties above is proved in the following lemma and proposition:

LEMMA 2.1.2. *There is a parallel unitary section $U \in C^\infty(M, \operatorname{End} E^+ \otimes \mathcal{A}_\infty)$ such that $U \mathcal{P}_0 U^* = \mathcal{P}_0^s = P_s$.*

PROOF. By the isomorphism $E^+ \otimes \mathcal{A} \cong S^+ \otimes (\mathcal{A}^+)^d \oplus S^- \otimes (\mathcal{A}^-)^d$ any $A \in M_{2d}(\mathcal{A})$ of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in M_d(\mathcal{A})$ defines a parallel section of $\operatorname{End} E^+ \otimes \mathcal{A}$.

By Lemma 1.4.3 there is a unitary $U \in M_{2d}(\mathcal{A}_\infty)$ of that form such that $U \mathcal{P}_0 U^* = \mathcal{P}_0^s = P_s$. $\qquad \square$

PROPOSITION 2.1.3. *For any $0 < b < \frac{1}{4}$ there is $W \in C^\infty(M, \operatorname{End}^+ E \otimes \mathcal{A}_\infty)$ with the properties above and such that W is parallel on $\{x \in M \mid d(x, \partial M) > b\}$.*

PROOF. By the previous lemma we may assume $\mathcal{P}_0 = P_s$.

In the following we identify $\partial M \times [0, b]$ with $\{x \in M \mid d(x, \partial M) \leq b\}$.

Since for $i = 1, 2$ the projection $\mathcal{P}_i$ is transverse to P_s, there are, by Lemma 1.4.4, smooth paths $W_i : [0, b] \to M_{2d}(\mathcal{A}_\infty)$ of unitaries with $[W_i, I_0] = 0$ such that $W_i(0) \mathcal{P}_i W_i^*(0) = \mathcal{P}_i^s$ and such that W_i is equal to the identity in a neighborhood of b and constant on $[0, \frac{b}{2}]$. They induce maps $\tilde{W}_i := \operatorname{id} \times W_i : (\partial_i M \cup \partial_{i+3} M) \times [0, b] \to M_{2d}(\mathcal{A}_\infty)$. The map

$$\cup_i (\tilde{W}_i \oplus \tilde{W}_i) : \partial M \times [0, b] \to M_{4d}(\mathcal{A}_\infty)$$

can be extended by 1 to a smooth section $W \in C^\infty(M, \operatorname{End}^+ E \otimes \mathcal{A}_\infty)$. By construction W has the right properties. $\qquad \square$

2.2. The operator D_I on $[0, 1]$

2.2.1. Definition and comparison with D_{I_s}. Let ∂ be the differentiation operator on $[0, 1]$.

For a pair $R = (P_0, P_1)$ of transverse Lagrangian projections with $P_j \in M_{2d}(\mathcal{A}_\infty)$, $j = 0, 1$, and $k \in \mathbb{N}_0 \cup \{\infty\}$ we define the function space

$$C_R^k([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$$
$$:= \{ f \in C^k([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d}) \mid P_x (I_0 \partial)^l f(x) = (I_0 \partial)^l f(x)$$
$$\text{for } x = 0, 1; \ l \in \mathbb{N}_0, \ l \leq k \}$$

and endow it with the subspace topology.

Given a pair (P_0, P_1) of transverse Lagrangian projections on $\mathcal{A}^{2d}$ we write D_I for the closure of the operator $I_0 \partial$ on the Hilbert $\mathcal{A}$-module $L^2([0, 1], \mathcal{A}^{2d})$ with domain $C_R^\infty([0, 1], \mathcal{A}^{2d})$.

We write D_{I_s} if $(P_0, P_1) = (P_s, 1 - P_s)$.

PROPOSITION 2.2.1. *Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$. Then for $0 < x_1 < x_2 < 1$ there is $U \in C^\infty([0, 1], M_{2d}(\mathcal{A}_\infty))$ such that*

    (1) $UU^* = 1$,
    (2) $UI_0 = I_0 U$,
    (3) $U(0)P_0 U(0)^* = P_s$,
    (4) $U(1)P_1 U(1)^* = 1 - P_s$,
    (5) U *is constant on* $[0, x_1]$ *and on* $[x_2, 1]$.

PROOF. By Lemma 1.4.3 there is a unitary $U_0 \in M_{2d}(\mathcal{A}_\infty)$ with $U_0 I_0 = I_0 U_0$ and $U_0 P_0 U_0^* = P_s$. Since $U_0 P_1 U_0^*$ is transverse to P_s, we can apply Lemma 1.4.4 to $P = U_0 P_1 U_0^*$ and $\tilde{P} = 1 - P_s$ in order to get a smooth path of unitaries $U_1 : [0, 1] \to M_{2d}(\mathcal{A}_\infty)$ such that $U(t) := U_1(t)U_0$ has the right properties. $\qquad\square$

PROPOSITION 2.2.2. *Let (P_0, P_1) be a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$ and let D_I be the associated operator.*

    (1) *Let U be as in the previous proposition. Then*

$$UD_I U^* = D_{I_s} + UI_0(\partial U^*)$$

    with $UI_0(\partial U^*) \in C^\infty([0, 1], M_{2d}(\mathcal{A}_\infty))$.
    (2) *The operator D_I is regular and selfadjoint.*

PROOF. (1) For $f \in C_R^\infty([0, 1], \mathcal{A}_\infty^{2d})$ with $R = (P_s, 1 - P_s)$ we have:

$$\begin{aligned} UD_I U^* f &= UI_0 \partial U^* f \\ &= D_{I_s} f + UI_0(\partial U^*)f \ . \end{aligned}$$

(2) follows as in Prop. 2.1.1. $\qquad\square$

2.2.2. Generalized eigenspace decomposition. We construct a decomposition of $L^2([0, 1], \mathcal{A}^{2d})$ into free finitely generated $\mathcal{A}$-modules preserved by D_I. For $d = 1$ these can be understood as analogues of eigenspaces.

Assume that the boundary conditions of D_I are given by a pair (P_0, P_1) with $P_0 = P_s$. For the general case use Lemma 1.4.3.

Let $p \in M_d(\mathcal{A})$ be such that $P_1 = \frac{1}{2}\begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix}$.

The transversality of P_0 and P_1 implies that $1 - p$ is invertible. Hence $\log p$ is defined for $\log : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$.

LEMMA 2.2.3. *Assume that $d = 1$.*
Then for $k \in \mathbb{Z}$ the function

$$f_k(x) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp[(-\tfrac{1}{2}\log p + \pi ki)x] + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\exp[(\tfrac{1}{2}\log p - \pi ki)x]\right)$$

is in $\operatorname{dom} D_I$ and $D_I f_k = \lambda_k f_k$ with $\lambda_k := -\tfrac{1}{2}i \log p - \pi k$.
The system $\{f_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis of the Hilbert $\mathcal{A}$-module $L^2([0, 1], \mathcal{A}^2)$.

Note that $\lambda_k f_k = f_k \lambda_k$ and $\sigma(\lambda_k) \subset]-\pi k, -\pi(k - 1)[$.

PROOF. Clearly the system $\{f_k\}_{k \in \mathbb{Z}}$ is orthonormal in the sense of Def. 5.1.17.
In order to prove that these functions form a basis we first show that there is an orthogonal projection onto the closure of their span. This implies that the span is orthogonally complemented. The second step will be to see that the complement

is trivial. Then the claim follows from Prop. 5.1.18.

The system $\{e_l^{\pm}(x) := v_{\pm}e^{2\pi ilx}\}_{l\in\mathbb{Z},\pm}$ with $v_+ := (1,0)$, $v_- := (0,1)$ is an orthonormal basis of $L^2([0,1],\mathcal{A}^2)$ by Prop. 5.1.18.

We have that

$$
\begin{aligned}
\langle f_k, e_l^{\pm}\rangle &= \frac{1}{\sqrt{2}}\int_0^1 \exp((\mp\tfrac{1}{2}\log p^* \mp \pi ki)x)\exp(2\pi ilx)\ dx \\
&= \frac{1}{\sqrt{2}}\left(\frac{(-1)^k\exp(\mp\tfrac{1}{2}\log p^*)-1}{\mp\tfrac{1}{2}\log p^* + \pi i(2l\mp k)}\right)\ ,
\end{aligned}
$$

hence $\{\langle f_k, e_l^{\pm}\rangle\}_{k\in\mathbb{Z}}\in l^2(\mathcal{A})$ for all $l\in\mathbb{Z}$ and for $\pm$. Thus

$$
Pe_l^{\pm} := \sum_{k\in\mathbb{Z}}f_k\langle f_k, e_l^{\pm}\rangle \in L^2([0,1],\mathcal{A}^2)\ .
$$

The $\mathcal{A}$-linear extension of P to the algebraic span of the functions $e_l^{\pm}$ has norm one. It follows that its closure is an orthogonal projection

$$
P: L^2([0,1],\mathcal{A}^2) \to \overline{\operatorname{span}_{\mathcal{A}}\{f_k \mid k\in\mathbb{Z}\}}\ .
$$

It remains to show that the kernel of P is trivial.

Let $g = (g_1, g_2)\in L^2([0,1],\mathcal{A}^2)$ with $\langle f_k, g\rangle = 0$ for all $k\in\mathbb{Z}$.

Hence for all $k\in\mathbb{Z}$

$$
(*)\qquad \int_0^1 \exp((-\tfrac{1}{2}\log p^* - \pi ki)x)g_1(x) + \exp((\tfrac{1}{2}\log p^* + \pi ki)x)g_2(x)\ dx = 0\ .
$$

Since $\exp(-\tfrac{1}{2}(\log p^*)\ x)g_1(x)$ is in $L^2([0,1],\mathcal{A})$, there is a unique $\{\lambda_l\}_{l\in\mathbb{Z}}\in l^2(\mathcal{A})$ such that

$$
\sum_{l\in\mathbb{Z}}\lambda_l e^{2\pi ilx} = \exp(-\tfrac{1}{2}(\log p^*)\ x)g_1(x)\ .
$$

Inserting this in $(*)$ and evaluating the integral for k even leads to

$$
\lambda_{k/2} + \int_0^1 \exp((\tfrac{1}{2}\log p^* + \pi ki)x)g_2(x)dx = 0\ .
$$

It follows that

$$
\exp(\tfrac{1}{2}(\log p^*)\ x)g_2(x) = \sum_{l\in\mathbb{Z}}(-\lambda_l)e^{-2\pi ilx}\ .
$$

Substituting again and evaluating $(*)$ for $k = 2\nu + 1$ with $\nu\in\mathbb{Z}$ we obtain

$$
\begin{aligned}
0 &= \int_0^1 \sum_{l\in\mathbb{Z}}\lambda_l(e^{\pi i(2(l-\nu)-1)x} - e^{-\pi i(2(l-\nu)-1)x})dx \\
&= -\sum_{l\in\mathbb{Z}}\lambda_l\frac{4}{\pi i(2(l-\nu)-1)}\ .
\end{aligned}
$$

Note that for every $l\in\mathbb{Z}$ the function

$$
a_l: \mathbb{Z}\to\mathbb{C},\ \nu\mapsto\frac{2}{\pi i(2(l-\nu)-1)}
$$

is in $l^2(\mathbb{C})$. We claim that $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$. Then $\{a_l\}_{l\in\mathbb{Z}}$ is an orthonormal basis of $l^2(\mathcal{A})$ as well, thus $\lambda_l = 0$ for all $l\in\mathbb{Z}$ and hence $g_1 = g_2 = 0$.

The Fourier transform of

$$a_0 : \nu \mapsto \frac{-2}{\pi i (2\nu + 1)}$$

is

$$h(x) = -ie^{-\pi ix}(1_{[0,1/2]}(x) - 1_{]1/2,1]}(x)) \in L^2(\mathbb{R}/\mathbb{Z}) \ .$$

Since $a_l(\nu) = a_0(\nu - l)$, the Fourier transform of a_l is $h(x)e^{-2\pi ilx}$.

By $hh^* = 1$ the system $\{h(x)e^{-2\pi ilx}\}_{l \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$. This implies that $\{a_l\}_{l \in \mathbb{Z}}$ is an orthonormal basis of $l^2(\mathbb{C})$.   □

For general d there is a decomposition of $L^2([0,1]\mathcal{A}^{2d})$ into $\mathcal{A}$-modules of rank d:

PROPOSITION 2.2.4. *For $k \in \mathbb{Z}$ let $U_k \subset L^2([0,1], \mathcal{A}^{2d})$ be the right $\mathcal{A}$-module spanned by the column vectors of*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \exp[(-\frac{1}{2}\log p + \pi ki)x] \\ \exp[(\frac{1}{2}\log p - \pi ki)x] \end{pmatrix} \in C^\infty([0,1], M_{2d \times d}(\mathcal{A}_\infty)) \ .$$

Each U_k is a free right $\mathcal{A}$-module of rank d and

$$L^2([0,1], \mathcal{A}^{2d}) = \widehat{\bigoplus_{k \in \mathbb{Z}}} U_k \ .$$

The sum is orthogonal.

Furthermore $U_k \subset \mathrm{dom}\, D_I$ and for $f \in U_k$

$$D_I f = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f \in U_k$$

with $\lambda_k = -\frac{1}{2}i\log p - \pi k$.

PROOF. The projections $P_0, P_1 \in M_{2d}(\mathcal{A})$ are Lagrangian projections on $M_d(\mathcal{A})^2$ (see remark after Def. 1.4.2).

Let $\tilde{D}_I$ be the closure of $I_0\partial$ on $L^2([0,1], M_d(\mathcal{A})^2)$ with domain $C_R^\infty([0,1], M_{2d}(\mathcal{A}))$ with $R = (P_0, P_1)$. Then $\tilde{D}_I = \oplus^d D_I$ with respect to the decomposition as right $M_d(\mathcal{A})$-modules

$$L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1], \mathcal{A}^{2d})^d$$

induced by the decomposition of a matrix into its column vectors.

By the previous proposition the Hilbert $M_d(\mathcal{A})$-module $L^2([0,1], M_d(\mathcal{A})^2)$ has an orthonormal basis $\{f_k\}_{k \in \mathbb{Z}}$ such that

$$\tilde{D}_I f_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} f_k \ .$$

For $k \in \mathbb{Z}$ let P_k be the orthogonal projection onto the span of f_k in $L^2([0,1], M_d(\mathcal{A})^2)$. It is diagonal with respect to the decomposition $L^2([0,1], M_d(\mathcal{A})^2) = L^2([0,1], \mathcal{A}^{2d})^d$.

Hence

$$L^2([0,1], \mathcal{A}^{2d}) = \widehat{\bigoplus_{k \in \mathbb{Z}}} P_k L^2([0,1], \mathcal{A}^{2d}) \ .$$

The assertion follows now since $P_k L^2([0,1], \mathcal{A}^{2d}) = U_k$. The module U_k is free since $\exp[(-\frac{1}{2}\log p + \pi ki)x]$ is invertible in $M_d(\mathcal{A})$ for all $x \in [0,1]$.   □

COROLLARY 2.2.5. *Let $\lambda \in \mathbb{C}$. The operator $D_I - \lambda$ has a bounded inverse on $L^2([0,1], \mathcal{A}^{2d})$ if and only if $\exp(2i\lambda) \notin \sigma(p)$.*

PROOF. By the previous proposition $(D_I - \lambda)|_{U_k}$ is invertible if and only if $e^{2i\lambda} \notin \sigma(p)$. Furthermore for λ with $e^{2i\lambda} \notin \sigma(p)$ the inverse of $(D_I - \lambda)|_{U_k}$ is uniformly bounded in k. Hence the closure of $\oplus_k (D_I - \lambda)|_{U_k}$ has a bounded inverse by Cor. 5.1.20. In particular the closure of $\oplus_k D_I|_{U_k}$ is selfadjoint. Since D_I is a selfadjoint extension of $\oplus_k D_I|_{U_k}$, it follows that D_I is the closure of $\oplus_k D_I|_{U_k}$. Hence $D_I - \lambda$ has a bounded inverse if $\exp(2i\lambda) \notin \sigma(p)$. $\square$

2.3. The operator D_Z on the cylinder

Let $X = \mathbb{R}, \mathbb{R}/\mathbb{Z}$. Endow $X \times [0,1]$ with the euclidean metric and a spin structure and let (x_1, x_2) be the euclidian coordinates of $X \times [0,1]$. Let $S = S^+ \oplus S^-$ be the spinor bundle on $X \times [0,1]$ endowed with the Levi-Cività connection and a parallel metric. Then $S \otimes \left((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d\right)$ is Dirac bundle on $X \times [0,1]$ with the $\mathcal{A}$-valued scalar product induced by the hermitian metric on S and the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^d$. Let $\bar{\partial}_Z$ be the associated Dirac operator.

We choose a parallel unit section s of S^+ and identify $S \otimes \left((\mathcal{A}^+)^d \oplus (\mathcal{A}^-)^d\right)$ with the trivial bundle $(X \times [0,1]) \times \left((\mathcal{A}^+)^{2d} \oplus (\mathcal{A}^-)^{2d}\right)$ via the isomorphisms

$$S_x^+ \otimes (\mathcal{A}^+)^d \oplus S_x^- \otimes (\mathcal{A}^-)^d \to (\mathcal{A}^+)^d \oplus (\mathcal{A}^+)^d \ ,$$

$$(s(x) \otimes v) \oplus (ic(dx_1)s(x) \otimes w) \mapsto (v, w) \ ,$$

and

$$S_x^- \otimes (\mathcal{A}^+)^d \oplus S_x^+ \otimes (\mathcal{A}^-)^d \to (\mathcal{A}^-)^d \oplus (\mathcal{A}^-)^d \ ,$$

$$(ic(dx_1)s(x) \otimes v) \oplus (s(x) \otimes w) \mapsto (v, w) \ .$$

for $x \in X \times [0,1]$.

Let $I := c(dx_2)c(dx_1) = I_0 \oplus (-I_0)$. Then

$$\bar{\partial}_Z = c(dx_1)(\partial_{x_1} + I\partial_{x_2}) \ .$$

Given a pair (P_0, P_1) of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_i \in M_{2d}(\mathcal{A}_\infty)$ we define D_Z to be the closure of $\bar{\partial}_Z$ with domain

$$\{f \in C_c^\infty(X \times [0,1], \mathcal{A}_\infty^{4d}) \mid (P_i \oplus P_i)f(x,i) = f(x,i) \text{ for all } x \in X, \ i = 0,1\}$$

on the Hilbert $\mathcal{A}$-module $L^2(X \times [0,1], \mathcal{A}^{4d})$.

We write $H(D_Z)$ for the Hilbert $\mathcal{A}$-module whose underlying right $\mathcal{A}$-module is $\operatorname{dom} D_Z$ and whose $\mathcal{A}$-valued scalar product is $\langle f, g \rangle_{D_Z} = \langle f, g \rangle + \langle D_Z f, D_Z g \rangle$ (see §5.1.3).

PROPOSITION 2.3.1.   (1) *The operator D_Z is selfadjoint on $L^2(X \times [0,1], \mathcal{A}^{4d})$ and has a bounded inverse.*

(2) *If $X = \mathbb{R}/\mathbb{Z}$, then the inclusion $\iota : H(D_Z) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$ is compact.*

PROOF. The proof is analogous to the family case [**BK**].

(1) By Lemma 1.4.3 we may assume $P_0 = P_s$. Recall from Prop. 2.2.4 that the operator D_I with boundary conditions (P_0, P_1) induces a decomposition

$$L^2([0,1], (\mathcal{A}^+)^{2d}) = \hat{\bigoplus_{l \in \mathbb{Z}}} U_l$$

such that for each $l \in \mathbb{Z}$ there is $\lambda_l \in M_d(\mathcal{A})$ with $D_I f = \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} f$ for $f \in U_l$.

We define submodules $U_{l,+} := U_l$ and $U_{l,-} := ic(dx_1)U_l$ of $L^2([0,1], \mathcal{A}^{4d})$. Then

$$\bigoplus_{l \in \mathbb{Z}} (U_{l,+} \oplus U_{l,-}) L^2(X)$$

is dense in $L^2(X \times [0,1], \mathcal{A}^{4d})$.

First consider the case $X = \mathbb{R}$:

For $l \in \mathbb{Z}$ let $\partial_{l,\pm}$ be the closure of the unbounded operator $(\partial_{x_1} \pm \lambda_l)$ on $U_{l,\pm} L^2(\mathbb{R})$ with domain $U_{l,\pm} \mathcal{S}(\mathbb{R})$ and let ∂_e be the closure of $c(dx_1)\left(\bigoplus_{l \in \mathbb{Z}, \pm} \partial_{l,\pm}\right)$.

The operator D_Z is an extension of ∂_e.

We claim that ∂_e has a bounded inverse on $L^2(\mathbb{R} \times [0,1], \mathcal{A}^{4d})$.

The Fourier transform on $L^2(\mathbb{R})$ induces an automorphism on $U_{l,\pm} L^2(\mathbb{R})$. Conjugation by it transforms $\partial_{l,\pm}$ into multiplication by

$$ix_1 \pm \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} .$$

Since $\sigma(\lambda_l) \subset \mathbb{R}^*$, we see that the operator $\partial_{l,\pm}$ has a bounded inverse and the norm of the inverse tends to zero for $l \to \pm\infty$. By Cor. 5.1.20 it follows that the closure of $\oplus_{l,\pm}\partial_{l,\pm}$ has a bounded inverse.

Hence the operator ∂_e has a bounded inverse as well. In particular it is selfadjoint, thus $D_Z = \partial_e$. For $X = \mathbb{R}$ the assertion follows.

If $X = \mathbb{R}/\mathbb{Z}$, then the spaces $U_{l,\pm} L^2(\mathbb{R}/\mathbb{Z})$ decompose further into a direct sum

$$U_{l,\pm} L^2(\mathbb{R}/\mathbb{Z}) = \widehat{\bigoplus_{k \in \mathbb{Z}}} V_{kl,\pm}$$

with $V_{kl,\pm} := e^{2\pi i k x_1} U_{l,\pm}$. Note that $V_{kl,\pm}$ is isomorphic to $\mathcal{A}^d$ as a Hilbert $\mathcal{A}$-module.

Let $\partial_{kl,\pm} \in B(V_{kl,\pm})$ be defined by

$$\partial_{kl,\pm} f := (\partial_{x_1} \pm \lambda_l) f = \begin{pmatrix} 2\pi i k \pm \lambda_l & 0 \\ 0 & 2\pi i k \pm \lambda_l \end{pmatrix} f$$

and let ∂_e be the closure of $c(dx_1)\left(\bigoplus_{k,l,\pm} \partial_{kl,\pm}\right)$.

The operator D_Z is an extension of ∂_e.

Since $|(2\pi i k \pm \lambda_l)^{-1}|$ tends to zero for $k, l \to \pm\infty$, the closure of $\oplus_{kl,\pm}\partial_{kl,\pm}$ has a compact inverse by Cor. 5.1.20 and hence ∂_e has a compact inverse as well.

Now (1) follows as above.

(2) follows from the compactness of D_Z^{-1} for $X = \mathbb{R}/\mathbb{Z}$ since $\iota = D_Z^{-1} D_Z : H(D_Z) \to L^2(\mathbb{R}/\mathbb{Z} \times [0,1], \mathcal{A}^{4d})$. $\square$

2.4. The index of D^+

Recall from §2.1.1 that the boundary conditions of D were defined by a triple $R = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$. For an open precompact subset U of M we define $H^1_{R0}(U, E \otimes \mathcal{A})$ to be the closure of $C^\infty_{Rc}(U, E \otimes \mathcal{A})$ in $H(D)$ (see §5.1.3 for the definition of $H(D)$).

Note that $H(D)$ is isomorphic to $l^2(\mathcal{A})$ as a Hilbert $\mathcal{A}$-module, since $L^2(M, E \otimes \mathcal{A})$ is isomorphic to $l^2(\mathcal{A})$ and since $L^2(M, E \otimes \mathcal{A})$ and $H(D)$ are isomorphic by Lemma 5.1.10.

LEMMA 2.4.1. *The inclusion* $\iota : H^1_{R0}(M_r, E \otimes \mathcal{A}) \to L^2(M, E \otimes \mathcal{A})$, $r \geq 0$, *is compact.*

PROOF. Let $\mathcal{V} := \{V_k\}_{k \in L}$ be an open covering of M_r such that the index set L is a finite subset of $\mathbb{N}$ with $1 \in L$ and such that:

- $V_1 = M \setminus \overline{F(r, \frac{1}{6})}$;
- for $k > 1$ there is an isometry $V_k \cong]0, \frac{1}{2}[\times [0, \frac{1}{5}[$. In particular V_k is in the flat region and has exactly one boundary component.

First we prove that the maps $\iota_k : H^1_{R0}(V_k, E \otimes \mathcal{A}) \to L^2(V_k, E \otimes \mathcal{A})$ are compact for all $k \in L$.

The compactness of ι_1 follows from the Sobolev embedding theorem ([**MF**], Lemma 3.3).

For $k \neq 1$ let $i \in \{0, 1, 2\}$ be such that $\partial V_k \subset (\partial_i M \cup \partial_{i+3} M)$ and set $P_k := \mathcal{P}_i$. Let D_k be the operator D_Z on the bundle $(\mathbb{R}/\mathbb{Z} \times [0, 1]) \times \mathcal{A}^{4d}$ with boundary conditions given by the pair $(P_k, 1 - P_k)$.

The map ι_k is compact since it factors through the map $H(D_k) \to L^2(\mathbb{R}/\mathbb{Z} \times [0, 1], \mathcal{A}^{4d})$, which is compact by Prop. 2.3.1.

Let $\{\phi_k\}_{k \in L}$ be a smooth partition of unity subordinate to the covering $\mathcal{V}$ such that $\partial_{e_2} \phi_k(x) = 0$ for all $x \in \partial M$ and $k \in L$. Multiplication with ϕ_k is a bounded map from $H^1_{R0}(M_r, E \otimes \mathcal{A})$ to $H^1_{R0}(V_k, E \otimes \mathcal{A})$, hence by $\iota = \sum_{k \in L} \iota_k \phi_k$, the inclusion ι is compact. $\qquad\square$

Let $H(D)^+$ be the subspace of $H(D)$ of homogeneous elements of degree zero.

PROPOSITION 2.4.2. *The operator*

$$D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$$

is a Fredholm operator.

PROOF. By constructing a parametrix for D^+ we show that D^+ is Fredholm (see Prop. 5.1.4).

The construction of the parametrix is analogous to the construction in the family case [**BK**]:

Choose a smooth partition of unity $\{\phi_k\}_{k \in J}$ subordinate to the covering $\mathcal{U}(0, \frac{1}{4})$ (defined in §1.1) and a system of smooth functions $\{\gamma_k\}_{k \in J}$ on M such that for all $k \in J$

- supp $\gamma_k \subset \mathcal{U}_k$ and $\gamma_k \phi_k = \phi_k$,
- $\partial_{e_2} \gamma_k(x) = \partial_{e_2} \phi_k(x) = 0$ for all $x \in \partial M$.

For $k \in \mathbb{Z}/6$ let D_{Z_k} be the operator defined in §2.3 on $L^2(Z_k, \mathcal{A}^{4d})$ with boundary conditions given by $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$. By Prop. 2.3.1 it is invertible. Let

$$Q_k := D_{Z_k}^{-1} : L^2(Z_k, \mathcal{A}^{4d}) \to H(D_{Z_k}) \ .$$

Since the symbol of D is elliptic and since $\mathcal{U}_\clubsuit$ is precompact, there is a parametrix $Q_\clubsuit : L^2(\mathcal{U}_\clubsuit, E \otimes \mathcal{A}) \to H^1_0(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$ such that $\gamma_\clubsuit (DQ_\clubsuit - 1)\phi_\clubsuit$ resp. $\gamma_\clubsuit (Q_\clubsuit D - 1)\phi_\clubsuit$ is compact on $L^2(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$ resp. $H^1_0(\mathcal{U}_\clubsuit, E \otimes \mathcal{A})$ [**MF**]. Furthermore $Q_\clubsuit$ can be chosen to be an odd operator.

We claim that

$$Q := \sum_{k \in J} \gamma_k Q_k \phi_k : L^2(M, E \otimes \mathcal{A}) \to H(D)$$

is a parametrix of D. Then it follows that $Q^- : L^2(M, E^- \otimes \mathcal{A}) \to H(D)^+$ is a parametrix of D^+.

In the following calculations the operators D_{Z_k} and the restriction of D to $\mathcal{U}_k$ are denoted by D as well. Let $\sim$ denote equality up to compact operators.

On $L^2(M, E \otimes \mathcal{A})$

$$DQ - 1 = \sum_{k \in J} [D, \gamma_k] Q_k \phi_k + \sum_{k \in J} \gamma_k D Q_k \phi_k - 1$$
$$\sim \sum_{k \in J} c(d\gamma_k) Q_k \phi_k .$$

Since $c(d\gamma_k) Q_k \phi_k$ is bounded from $L^2(M, E \otimes \mathcal{A})$ to $H^1_{R0}(M_r, E \otimes \mathcal{A})$ with $r > 0$ big enough, it is a compact operator on $L^2(M, E \otimes \mathcal{A})$ by the previous lemma. Hence $DQ - 1$ is compact.

On $H(D)$

$$QD - 1 = \sum_{k \in J} \gamma_k Q_k [D, \phi_k] + \sum_{k \in J} \gamma_k Q_k D\phi_k - 1$$
$$\sim \sum_{k \in J} \gamma_k Q_k c(d\phi_k) .$$

Here $c(d\phi_k) : H(D) \to L^2(\mathcal{U}_k, E \otimes \mathcal{A})$ is a compact operator by the previous lemma since $\operatorname{supp} c(d\phi_k)$ is compact. Moreover $\gamma_k Q_k : L^2(\mathcal{U}_k, E \otimes \mathcal{A}) \to H(D)$ is bounded, hence $QD - 1 : H(D) \to H(D)$ is compact. $\square$

LEMMA 2.4.3. *Let* $P_0, P_1, P_2 : [0, 1] \to M_{2d}(\mathcal{A})$ *be continuous paths of Lagrangian projections and assume that* $P_0(t), P_1(t), P_2(t)$ *are pairwise transverse for each* $t \in [0, 1]$. *Let* $R(t) = (P_0(t), P_1(t), P_2(t))$. *We define* $D(t)$ *to be the closure of* $\partial\!\!\!/_E$ *on* $L^2(M, E \otimes \mathcal{A})$ *with domain* $C^\infty_{R(t)c}(M, E \otimes \mathcal{A})$.

Let $\operatorname{ind} D(t)^+$ *be the index of the Fredholm operator* $D(t)^+ : H(D(t))^+ \to L^2(M, E \otimes \mathcal{A})$.

Then

$$\operatorname{ind} D(0)^+ = \operatorname{ind} D(1)^+ \in K_0(\mathcal{A}) .$$

PROOF. There is a continuous path of unitaries $[0, 1] \to C^\infty(M, \operatorname{End} E^+ \otimes \mathcal{A})$, $t \mapsto W_t$, such that $W_t D(t)^+ W_t^* = D_s^+ + W_t c(dW_t^*)$. The family

$$D_s^+ + W_t c(dW_t^*) : H(D_s)^+ \to L^2(M, E^- \otimes \mathcal{A})$$

is a continuous path of Fredholm operators, thus $\operatorname{ind} W_0 D(0)^+ W_0^* = \operatorname{ind} W_1 D(1)^+ W_1^*$ by Prop. 5.1.7. $\square$

PROPOSITION 2.4.4. *The index of* $D^+ : H(D)^+ \to L^2(M, E^- \otimes \mathcal{A})$ *is*

$$\operatorname{ind} D^+ = \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) \in K_0(\mathcal{A}) .$$

PROOF. The argument is analogous to the one in [**BK**].

By Lemma 2.1.2 we may assume $\mathcal{P}_0 = P_s$.

Let $a_j := i \frac{p_j + 1}{p_j - 1} \in M_{2d}(\mathcal{A})$. Then $\mathcal{P}_j = P(a_j)$, $j = 1, 2$, in the notation of §1.4.2 .

Let $p^+ := 1_{\{x > 0\}}(a_1 - a_2)$.

From Lemma 1.4.8 it follows that

$$\tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [p^+] - [1 - p^+] \in K_0(\mathcal{A})$$

and that there are continuous paths $P_1, P_2 : [0, 2] \to M_{2d}(\mathcal{A})$ of Lagrangian projections with $P_j(0) = \mathcal{P}_j$, $j = 1, 2$, with $P_1(2) = P(2p^+ - 1)$ and $P_2(2) = P(1 - 2p^+)$ and such that $P_s, P_1(t), P_2(t)$ are pairwise transverse for all $t \in [0, 2]$.

For $t \in [0, 2]$ let $D(t)$ be the Dirac operator on $L^2(M, E \otimes \mathcal{A})$ whose boundary conditions are given by the triple $(P_s, P_1(t), P_2(t))$. The previous lemma implies that $\operatorname{ind} D(0)^+ = \operatorname{ind} D(2)^+$.

We show that the index of $D(2)^+$ equals $[p^+] - [1 - p^+]$:

Let $Q_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{C})$, $Q_1 = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $Q_2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Then

$$P_1(2) = (Q_1 \otimes p^+) \oplus (Q_2 \otimes (1 - p^+))$$

and

$$P_2(2) = (Q_2 \otimes p^+) \oplus (Q_1 \otimes (1 - p^+))$$

with respect to the decomposition

$$E^+ \otimes \mathcal{A} = (S \otimes p^+ \mathcal{A}^n) \oplus (S \otimes (1 - p^+)\mathcal{A}^n) \ .$$

The Dirac operator respects the decomposition. By [**BK**] the Dirac operator associated to the bundle $S \otimes (\mathbb{C}^+ \oplus \mathbb{C}^-)$ has index 1 if the boundary conditions are given by the triple (Q_0, Q_1, Q_2), and index -1 if they are are given by (Q_0, Q_2, Q_1). It follows that

$$\operatorname{ind} D(2)^+ = [p^+] - [1 - p^+] \ .$$

$\square$

2.5. A perturbation with closed range

Imitating the construction in [**BGV**], §9.5, we define a perturbation of D by a compact operator in order to obtain an operator with closed range. Then we can express the index of D^+ in terms of the kernel and cokernel of the perturbed operator.

Choose an orthonormal basis $\{\psi_i\}_{i \in \mathbb{N}} \subset L^2(M, E^-)$ such that $\psi_i \in C_c^\infty(M, E^-)$ and $\operatorname{supp} \psi_i \subset M \setminus \partial M$ for all $i \in \mathbb{N}$. By Prop. 5.1.18 this is an orthonormal basis of $L^2(M, E^- \otimes \mathcal{A})$ as well.

Since D^+ is a Fredholm operator, there is a projective $\mathcal{A}$-module $P \subset L^2(M, E^- \otimes \mathcal{A})$ and a closed $\mathcal{A}$-module $Q \subset \operatorname{Ran} D^+$ such that $P \oplus Q = L^2(M, E^- \otimes \mathcal{A})$. By Prop. 5.1.21 there is $\mathcal{N} \in \mathbb{N}$ such that $L_\mathcal{N} := \operatorname{span}_\mathcal{A}\{\psi_i \mid i = 1, \dots, \mathcal{N}\}$ fulfills $L_\mathcal{N} + P = L^2(M, E^- \otimes \mathcal{A})$. In particular it follows that

$$L_\mathcal{N} + \operatorname{Ran} D^+ = L^2(M, E^- \otimes \mathcal{A}) \ .$$

Let $M' := M \cup *$ be the disjoint union of M and one isolated point. Let E'^+ be the hermitian vector bundle $E^+ \cup (* \times \mathbb{C}^\mathcal{N})$ on M', where we endow $\mathbb{C}^\mathcal{N}$ with the standard hermitian product. Let E'^- be the hermitian bundle $E^- \cup (* \times \{0\})$ and let $E' = E'^+ \oplus E'^-$. Extend D by zero to a selfadjoint odd operator D' on $L^2(M', E' \otimes \mathcal{A})$.

As D is regular, D' is regular as well.

Furthermore $D' : H(D') \to L^2(M', E' \otimes \mathcal{A})$ is a Fredholm operator and

$$\operatorname{ind} D'^+ = \operatorname{ind} D^+ + [\mathcal{A}^\mathcal{N}] \ .$$

We extend the standard basis of $\mathbb{C}^{\mathcal{N}}$ by zero to a system $\{e_k\}_{k=1,\ldots,\mathcal{N}}$ of sections of E'^+ and define a compact selfadjoint odd operator K on $L^2(M', E' \otimes \mathcal{A})$ by

$$Kf := \sum_{k=1}^{\mathcal{N}} e_k \langle \psi_k, f \rangle + \psi_k \langle e_k, f \rangle \ .$$

Set $D(\rho) := D' + \rho K$ for $\rho \in \mathbb{R}$. Then

$$\operatorname{ind} D(\rho)^+ = \operatorname{ind} D'^+ = \operatorname{ind} D^+ + [\mathcal{A}^{\mathcal{N}}] \ .$$

Furthermore $D(\rho)$ is regular by Prop. 5.1.11.

By construction $D(\rho)^+$ is surjective for $\rho \neq 0$. Hence by Prop. 5.1.13 and Prop. 5.1.6 its kernel is a projective submodule of $H(D')^+$ and the kernel of $D(\rho)^-$ is trivial.

Hence:

PROPOSITION 2.5.1. *For $\rho \in \mathbb{R}$ the operator $D(\rho)^+$ is a Fredholm operator with index independent of ρ. For $\rho \neq 0$*

$$\operatorname{ind} D^+ = \operatorname{ind} D(\rho)^+ - [\mathcal{A}^{\mathcal{N}}] = [\operatorname{Ker} D(\rho)] - [\mathcal{A}^{\mathcal{N}}] \ .$$

From now on we write D, E, M for D', E', M' and we extend the operator D_s by zero to the new manifold M. Furthermore we redefine the open covering $\mathcal{U}(r, b)$ from §1.1 by including the isolated point in the set $\mathcal{U}_{\clubsuit}$.

CHAPTER 3

Heat Semigroups and Kernels

3.1. Complex heat kernels

In this section we collect some properties of the heat kernels associated to the operators D_{I_s} and D_s, which were defined in §2.2.1 and §2.1.2, and prove some technical lemmas for further reference. The results are proved by applying standard methods of the theory of partial differential equations to the particular geometric situation. The reader might skip this section at first reading.

3.1.1. The heat kernel of $e^{-tD_{I_s}^2}$. Since D_{I_s} is selfadjoint on $L^2([0,1], \mathbb{C}^{2d})$, the operator $-D_{I_s}^2$ generates a semigroup $e^{-tD_{I_s}^2}$ on $L^2([0,1], \mathbb{C}^{2d})$. In this section we determine the corresponding family of integral kernels by using the method of images (see [**Ta**], Ch. 3, §7) and study its properties.

The space $L^2([0,1], \mathbb{C}^{2d})$ decomposes into an orthogonal sum

$$L^2([0,1], P_s\mathbb{C}^{2d}) \oplus L^2([0,1], (1-P_s)\mathbb{C}^{2d})$$

and the semigroup $e^{-tD_{I_s}^2}$ is diagonal with respect to this decomposition.

We define an embedding

$$\tilde{\ } : L^2([0,1], \mathbb{C}^{2d}) \to L^2(\mathbb{R}/4\mathbb{Z}, \mathbb{C}^{2d})$$

by requiring that $\tilde{\ }$ is a right inverse of the map

$$L^2(\mathbb{R}/4\mathbb{Z}, \mathbb{C}^{2d}) \to L^2([0,1], \mathbb{C}^{2d}), \ f \mapsto f|_{[0,1]}$$

and that the image of $L^2([0,1], P_s\mathbb{C}^{2d})$ resp. $L^2([0,1], (1-P_s)\mathbb{C}^{2d})$ consists of functions that are even resp. odd with respect to $y = 0$ and $y = 2$ and odd resp. even with respect to $y = 1$ and $y = 3$.

Then $\tilde{\ }$ maps $C_R^\infty([0,1], \mathbb{C}^{2d})$ with $R = (P_s, 1 - P_s)$ into $C^\infty(\mathbb{R}/4\mathbb{Z}, \mathbb{C}^{2d})$ since

$$C_R^\infty([0,1], \mathbb{C}^{2d}) = C_l^\infty([0,1], P_s\mathbb{C}^{2d}) \oplus C_r^\infty([0,1], (1-P_s)\mathbb{C}^{2d}) .$$

Here

$\quad C_l^\infty([0,1], P_s\mathbb{C}^{2d})$
$\quad\quad := \ \{f \in C^\infty([0,1], P_s\mathbb{C}^{2d}) \mid (i\partial)^{2k} f(1) = 0, \ (i\partial)^{2k+1} f(0) = 0 \ \forall k \in \mathbb{N}_0\}$

and

$C_r^\infty([0,1], (1-P_s)\mathbb{C}^{2d})$
$\quad := \ \{f \in C^\infty([0,1], (1-P_s)\mathbb{C}^{2d}) \mid (i\partial)^{2k} f(0) = 0, \ (i\partial)^{2k+1} f(1) = 0 \ \forall k \in \mathbb{N}_0\} .$

The scalar heat kernel of $e^{t\partial^2}$ on $\mathbb{R}/4\mathbb{Z}$ is

$$H(t, x, y) = (4\pi t)^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} e^{-\frac{(x-y+4k)^2}{4t}} \ .$$

For $f \in L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ and $x \in [0,1]$

$$
\begin{aligned}
(e^{-tD_{I_s}^2}f)(x) &= (e^{t\partial^2}\tilde{f})(x) \\
&= \int_0^1 H(t,x,y)f(y)dy + \int_1^2 H(t,x,y)f(2-y)dy \\
&\quad + \int_2^3 H(t,x,y)(-f(y-2))dy + \int_3^4 H(t,x,y)(-f(4-y))dy .
\end{aligned}
$$

It follows that the action of $e^{-tD_{I_s}^2}$ on the space $L^2([0,1],(1-P_s)\mathbb{C}^{2d})$ is given by the scalar integral kernel

$$
(x,y) \mapsto H(t,x,y) + H(t,x,2-y) - H(t,x,y+2) - H(t,x,4-y) .
$$

Analogously we conclude that the action of $e^{-tD_{I_s}^2}$ restricted to $L^2([0,1],P_s\mathbb{C}^{2d})$ is given by the integral kernel

$$
(x,y) \mapsto H(t,x,y) - H(t,x,2-y) - H(t,x,y+2) + H(t,x,4-y) .
$$

This yields the integral kernel k_t of $e^{-tD_{I_s}^2}$.

In the following we write $C_R^\infty([0,1],M_{2d}(\mathbb{C}))$ for the space of functions in $C^\infty([0,1],M_{2d}(\mathbb{C}))$ with column vectors in $C_R^\infty([0,1],\mathbb{C}^{2d})$.

LEMMA 3.1.1. *The map*

$$
(0,\infty) \to C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C}))), \ t \mapsto \big(y \mapsto k_t(\cdot,y)\big) ,
$$

is smooth.

For $\phi,\psi \in C^\infty([0,1])$ with $\mathrm{supp}\,\phi \cap \mathrm{supp}\,\psi = \emptyset$ the map $t \mapsto \big(y \mapsto \phi(\cdot)k_t(\cdot,y)\psi(y)\big)$ can be extended by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1],C_R^\infty([0,1],M_{2d}(\mathbb{C})))$.

PROOF. This follows from the corresponding properties of H. $\square$

LEMMA 3.1.2. *Let $m,n \in \mathbb{N}_0$. Then there is $C > 0$ such that for all $x,y \in [0,1]$ and all $t > 0$*

$$
|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})e^{-\frac{d(x,y)^2}{4t}} .
$$

PROOF. The assertion follows from the explicit formula of H above. When estimating the derivatives we take into account that for all $m \in \mathbb{N}$ the function $(x,y,t) \mapsto \frac{(x-y)^{2m}}{t^m}e^{-\frac{(x-y)^2}{4t}}$ can be continuously extended by zero to $t = 0$. $\square$

3.1.2. The heat kernel of $e^{-tD_s^2}$. The operator D_s is selfadjoint on the Hilbert space $L^2(M,E)$. Hence $-D_s^2$ generates a semigroup on $L^2(M,E)$. In this section we prove the existence of the integral kernel of $e^{-tD_s^2}$ and study its properties.

In parallel we study the semigroup $e^{-tD_Z^2}$ on $L^2(Z,\mathbb{C}^{4d})$ where $Z = \mathbb{R} \times [0,1]$ and D_Z is the operator defined in §2.3. Here we assume that the boundary conditions are given by a pair (P_0,P_1) with $P_0,P_1 \in M_{2d}(\mathbb{C})$. Then D_Z is selfadjoint on $L^2(Z,\mathbb{C}^{4d})$. We will compare $e^{-tD_s^2}$ on the cylindric ends with $e^{-tD_Z^2}$ with appropriate boundary conditions in order to get estimates for the integral kernel of $e^{-tD_s^2}$.

Since the proofs are standard, they are only sketched here.

Recall that a solution $u : \mathbb{R} \to \mathrm{dom}\,D_s$ of the initial-value problem

$$
\frac{d}{dt}u(t) = iD_s u(t), \ u(0) = f
$$

with $f \in C^\infty_{Rc}(M, E)$ is unique by an energy estimate. An analogous statement holds for a solution of the corresponding problem for D_Z.

LEMMA 3.1.3. *If* $f \in C^\infty_{Rc}(M, E)$*, then* $d(x, \operatorname{supp} f) \leq |t|$ *for any* $x \in \operatorname{supp}(e^{itD_s}f)$*. An analogous result holds for* D_Z *on* $L^2(Z, \mathbb{C}^{4d})$*.*

This property is called "finite propagation speed property".

PROOF. The proof relies on a cutting-and-pasting argument.

For $j \in \mathbb{Z}/6$ let $V_j := \{x \in M \mid d(x, \partial_j M) < \frac{1}{4}\}$ and let $W := M \setminus \partial M$. These sets define an open covering of M. We show that the finite propagation speed property holds on these sets for small times. From this we conclude that it holds on M for all times.

By an oriented isometry we identify V_j with $\{0 \leq x_2 < \frac{1}{4}\} \subset \mathbb{R}^2$. Recall that $E|_{V_j}$ was identified with the trivial bundle with fiber $\mathbb{C}^{4d}$ in §1.1.

The restriction of $\partial\!\!\!/_E$ to V_j extends to a translation invariant differential operator $\partial\!\!\!/_{\mathbb{R}^2} : C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d}) \to L^2(\mathbb{R}^2, \mathbb{C}^{4d})$. Let $D_{\mathbb{R}^2}$ be the closure of $\partial\!\!\!/_{\mathbb{R}^2}$.

We define an embedding

$$\sim \; : C^\infty_{Rc}(V_j, \mathbb{C}^{4d}) \hookrightarrow C^\infty_c(\mathbb{R}^2, \mathbb{C}^{4d})$$

intertwining the operators D_s and $D_{\mathbb{R}^2}$ using the method of images similar to §3.1.1.

Recall that the triple defining the boundary conditions of D_s was denoted by $(\mathcal{P}^s_0, \mathcal{P}^s_1, \mathcal{P}^s_2)$. Let

$$C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3}\mathbb{C}^{2d})$$
$$:= \{f \in C^\infty_c(V_j, \mathcal{P}^s_{j \bmod 3}\mathbb{C}^{2d}) \mid (\partial^{2k+1}_{e_2}f)(x) = 0, \; \forall k \in \mathbb{N}_0, \; \forall x \in \partial_j M\}$$

and

$$C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3})\mathbb{C}^{2d})$$
$$:= \{f \in C^\infty_c(V_j, (1 - \mathcal{P}^s_{j \bmod 3})\mathbb{C}^{2d}) \mid (\partial^{2k}_{e_2}f)(x) = 0, \; \forall k \in \mathbb{N}_0, \; \forall x \in \partial_j M\} \;.$$

Then $C^\infty_{Rc}(V_j, E^+)$ and $C^\infty_{Rc}(V_j, E^-)$ decompose into a direct sum

$$C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3}\mathbb{C}^{2d}) \oplus C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3})\mathbb{C}^{2d}) \;.$$

For $f \in C^\infty_l(V_j, \mathcal{P}^s_{j \bmod 3}\mathbb{C}^{2d})$ resp. $f \in C^\infty_r(V_j, (1 - \mathcal{P}^s_{j \bmod 3})\mathbb{C}^{2d})$ we define $\tilde{f}$ by first extending f by zero to the half plane $\{x_2 \geq 0\}$ and then reflecting such that $\tilde{f}$ is even resp. odd with respect to $\{x_2 = 0\}$.

For $D_{\mathbb{R}^2}$ the finite propagation speed property holds. Hence the assertion of the lemma holds for all $f \in C^\infty_{Rc}(V_j, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial_j M) < \frac{3}{16}\}$ and for $|t| < \frac{1}{16}$.

For $f \in C^\infty_c(W, E)$ with $\operatorname{supp} f \subset \{x \in M \mid d(x, \partial M) > \frac{1}{16}\}$ and for $|t| < \frac{1}{16}$ the assertion holds by the standard theory of hyperbolic equations on open subsets of $\mathbb{R}^2$.

Since every $f \in C^\infty_c(M, E)$ can be written as $f = f_W + f_0 + \ldots f_5$ with $f_W \in C^\infty_c(W, E)$ and $f_j \in C^\infty_{Rc}(V_j, E)$, the assertion holds for every $f \in C^\infty_c(M, E)$ and for $|t| < \frac{1}{16}$, and by the group property of e^{itD_s} it follows for all $t \in \mathbb{R}$.

The proof for D_Z is analogous. $\qquad\square$

For $k \in \mathbb{N}_0$ let $H^k(\mathbb{C}, D_s)$ be the Hilbert space whose underlying vector space is $\operatorname{dom} D_s^k$ and whose scalar product is given by

$$\langle f, g \rangle_{H^k} := \langle (1 + D_s^2)^{\frac{k}{2}} f, (1 + D_s^2)^{\frac{k}{2}} g \rangle \ .$$

Define $H^k(\mathbb{C}, D_Z)$ analogously.

LEMMA 3.1.4. *Let* $k \in \mathbb{N}$, $k \geq 2$.
  (1) *There is an embedding* $H^k(\mathbb{C}, D_Z) \to C^{k-2}(Z, \mathbb{C}^{4d})$.
  (2) *There is an embedding* $H^k(\mathbb{C}, D_s) \to C^{k-2}(M, E)$.

PROOF. We sketch the proof of (2), the proof of (1) is analogous.

Let $D_{\mathbb{R}^2}$ be as in the previous proof.

For fixed $r > 0$ the constants in the Gårding inequality for the elliptic operator $(1 + D_{\mathbb{R}^2}^2)^k$ on balls $B_r(x)$, $x \in \mathbb{R}^2$, can be chosen independent of x.

For $j \in \mathbb{Z}/6$ the embedding $C_{Rc}^\infty(V_j, E) \hookrightarrow C_c^\infty(\mathbb{R}^2, \mathbb{C}^{4d})$ defined in the previous proof intertwines the Dirac operators D_s on $C_{Rc}^\infty(V_j, E)$ and $D_{\mathbb{R}^2}$ on $C_c^\infty(\mathbb{R}^2, \mathbb{C}^{4d})$. Hence the Gårding inequality for the operator $(1+D_s^2)^k$ on balls $B_{1/8}(x) \subset M$ with $x \in \partial M$ holds with constants independent of x.

For $r > 0$ fixed and small enough we can also find global constants for the Gårding inequality for $(1 + D_s^2)^k$ on balls $B_r(x) \subset M$ with $B_r(x) \cap \partial M = \emptyset$ since M is of bounded geometry.

Then the assertion follows from the Sobolev embedding theorem.   □

COROLLARY 3.1.5. *The operators* $e^{-tD_Z^2}$ *on* $L^2(Z, \mathbb{C}^{4d})$ *and* $e^{-tD_s^2}$ *on* $L^2(M, E)$ *are integral operators with smooth integral kernels.*

PROOF. This follows from the previous two lemmas (see [**Ro**], Lemma 5.6).   □

LEMMA 3.1.6. *Let* $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be a function and assume that for every* $\varepsilon, \delta > 0$ *there is* $C > 0$ *such that for all* $r > \varepsilon$ *and* $t > 0$

$$f(r, t) \leq C e^{-\frac{(r - \varepsilon/2)^2}{(4+\delta)t}} \ .$$

Then for all $\varepsilon, \delta > 0$ *there is* $C > 0$ *such that for all* $r > \varepsilon$ *and* $t > 0$

$$f(r, t) \leq C e^{-\frac{r^2}{(4+\delta)t}} \ .$$

PROOF. Choose $0 < a < 1$ with $\frac{1-a}{4+\delta/2} > \frac{1}{4+\delta}$ and let $m > \frac{2}{a}$.

Then there is $C > 0$ such that for all $r > \varepsilon$ and $t > 0$

$$f(r, t) \leq C e^{-\frac{(r - \varepsilon/m)^2}{(4+\delta/2)t}} \ .$$

It follows that

$$
\begin{aligned}
f(r, t) &\leq C e^{-\frac{(1-a)r^2}{(4+\delta/2)t}} e^{\frac{r}{(4+\delta/2)t}(-ar + \frac{2\varepsilon}{m})} e^{-\frac{(\varepsilon/m)^2}{(4+\delta/2)t}} \\
&\leq C e^{-\frac{r^2}{(4+\delta)t}} \ .
\end{aligned}
$$

In the last step we used the fact that $\frac{r}{(4+\delta/2)t}(-ar + \frac{2\varepsilon}{m}) < 0$ for $r > \varepsilon$.   □

LEMMA 3.1.7. *Let* N *be closed manifold resp. let* $N = M, Z$. *If* N *is a closed manifold, let* E_N *be a Dirac bundle on* N *and let* D_N *be the associated Dirac operator. If* $N = M$ *resp.* $N = Z$, *then let* $E_N = E$ *resp.* $E_N = Z \times \mathbb{C}^{4d}$ *and let* $D_N = D_s$ *resp.* $D_N = D_Z$. *Let* k_t *be the integral kernel of* $e^{-tD_N^2}$.

For every $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $t > 0$, $r > \varepsilon$ and $x \in N$

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy \leq Ce^{-\frac{r^2}{(4+\delta)t}} \ .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

PROOF. Let $k \in 2\mathbb{N}$ with $k > \frac{\dim N}{2}$.

Let $S(x, \varepsilon) := \{u \in C_c^\infty(B_\varepsilon(x), E_N) \mid \|(1 + D_N^2)^{-\frac{k}{2}} u\| \leq 1\}$.

Then by the Sobolev embedding theorem resp. by Lemma 3.1.4 there is $C > 0$ such that for all $x \in N$, $t > 0$ and $r > \varepsilon$

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy \leq C \sup_{u \in S(x,\varepsilon/2)} \|e^{-tD_N^2} u\|_{N \setminus B_r(x)}^2 \ .$$

By a standard argument using the finite propagation speed property of D_N (see the proof of [**CGT**], Prop. 1.1) it follows that

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy \quad \leq \quad Ct^{-1/2} \int_{r-\varepsilon/2}^{\infty} |(1 + (\frac{d}{ids})^2)^{k/2} e^{-s^2/4t}| \, ds$$

$$= \quad C \int_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} |(1 + t^{-1}(\frac{d}{ids'})^2)^{k/2} e^{-s'^2/4}| \, ds' \ .$$

Thus there is $l \in \mathbb{N}$ such that

$$\int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy \quad \leq \quad C(1 + t^{-l}) \int_{\frac{r-\varepsilon/2}{\sqrt{t}}}^{\infty} (1 + s'^l) e^{-s'^2/4} \, ds'$$

$$\leq \quad C(1 + t^{-l}) e^{-\frac{(r-\varepsilon/2)^2}{(4+\delta/2)t}}$$

$$\leq \quad Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} \ .$$

Then the assertion follows by applying the previous lemma to

$$f(r,t) := \sup_{x \in N} \int_{N \setminus B_r(x)} |k_t(x,y)|^2 \, dy \ .$$

For the derivatives the argument is similar. $\square$

LEMMA 3.1.8. *Let k_t be as in the previous lemma.*

For any $\varepsilon, \delta > 0$ there is $C < \infty$ such that for all $x, y \in N$ with $d(x,y) > \varepsilon$ and all $t > 0$

$$|k_t(x,y)| \leq Ce^{-\frac{d(x,y)^2}{(4+\delta)t}} \ .$$

Analogous estimates hold for the partial derivatives in x and y with respect to unit vector fields on N.

PROOF. Let $S(y, \varepsilon)$ and $k \in 2\mathbb{N}$ be as in the proof of the previous lemma. By the Sobolev embedding theorem resp. Lemma 3.1.4 there is $C > 0$ such that for all $r > \varepsilon$, $t > 0$ and all $x, y \in N$ with $d(x,y) \geq r$

$$|k_t(x,y)| \leq C \sup_{u \in S(y,\varepsilon/4)} \|(1 + D_N^2)^{\frac{k}{2}} e^{-tD_N^2} u\|_{N \setminus B_{r-\varepsilon/4}(y)}^2 \ .$$

As in the proof of the previous lemma this implies

$$|k_t(x,y)| \leq Ce^{-\frac{(r-\varepsilon/2)^2}{(4+\delta)t}} \ .$$

Then the assertion follows from Lemma 3.1.6 with $f(r,t) := \sup\limits_{x,y\in N:\ d(x,y)=r} |k_t(x,y)|.$

For the derivatives the argument is similar. $\square$

For the next lemma assume that $U \subset M$ is an open set for which one of the following properties holds:

(1) U is precompact and $\overline{U} \cap \partial M = \emptyset$,
(2) there is $k \in \mathbb{Z}/6$ such that $U \subset F_k(0, \frac{1}{4})$.

In the first case there is a closed manifold N and a Dirac bundle E_N on N such that there is a Dirac bundle isomorphism $E|_U \to E_N$ whose base map is an isometry. We identify $E|_U$ with its image in E_N. Then D_s coincides with D_N on U.

In the second case U is a subset of Z_k by §1.1. Let D_{Z_k} be the operator D_Z on $Z_k \times \mathbb{C}^{4d}$ with boundary conditions given by $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$. Then D_s coincides with D_{Z_k} on U.

LEMMA 3.1.9. *Let U be as in (1) resp. (2). Let k_t be the integral kernel of the heat semigroup of D_s on M and let k'_t be the integral kernel of the heat semigroup of D_N resp. D_{Z_k}.*

For every $T > 0$ and $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $0 < t < T$, $r > \varepsilon$ and $x, y \in U$ with $B_r(x), B_r(y) \subset U$

$$|k_t(x,y) - k'_t(x,y)| \leq Ce^{-\frac{r^2}{(4+\delta)t}} \ .$$

Analogous estimates hold for the partial derivatives with respect to unit vector fields on U.

PROOF. The notation is as in the proof of Lemma 3.1.7.
The estimate follows from

$$
\begin{aligned}
|k_t(x,y) - k'_t(x,y)| \quad &= \quad \sup_{\phi \in S(x,\varepsilon)} \sup_{\psi \in S(y,\varepsilon/2)} |\langle \phi, e^{-tD_s^2}\psi \rangle - \langle \phi, e^{-tD_N^2}\psi \rangle| \\
&\leq \quad Ct^{-\frac{1}{2}} \sup_{\phi \in S(x,\varepsilon)} \sup_{\psi \in S(y,\varepsilon/2)} |\langle \phi, \int_{\mathbb{R}} e^{-s^2/4t}(e^{isD_s} - e^{isD_N})\psi \rangle| \\
&\leq \quad Ct^{-\frac{1}{2}} \int\limits_{r-\varepsilon/2}^{\infty} |(1 + (\frac{d}{ids})^2)^k e^{-s^2/4t}| \ .
\end{aligned}
$$

Here we used that $(e^{isD_s} - e^{isD_N})\psi = 0$ for $|s| \leq r - \varepsilon/2$ by the finite propagation speed property (Lemma 3.1.3) and the uniqueness of solutions of hyperbolic equations. $\square$

3.2. The heat semigroup on closed manifolds

Let $\mathcal{B}$ be a Banach algebra with unit.
Let N be a closed manifold of dimension n. Let E_N be a Dirac bundle on N and let D_N be the associated Dirac operator. For simplicity (we will only need this case) assume that E_N is trivial as a vector bundle.

By Cor. 5.2.4 the associated heat kernel defines a family of bounded operators on $L^2(N, E_N \otimes \mathcal{B})$. The operators are smoothing, thus they restrict to a family of bounded operators on $C^m(N, E_N \otimes \mathcal{B})$ for any $m \in \mathbb{N}_0$. In order to show that the family extends to a holomorphic semigroup we have to study its behavior for small times.

By Lemma 5.2.10 we can define $-D_N^2$ as a closed operator on $L^2(N, E_N \otimes \mathcal{B})$ by requiring that $C^\infty(N, E_N \otimes \mathcal{B})$ is a core of $-D_N^2$.

For $t \to 0$ the heat kernel $k_t \in C^\infty(N \times N, E_N \boxtimes E_N)$ can be estimated as follows:

LEMMA 3.2.1. *Let $\varepsilon > 0$ be smaller than the injectivity radius of N and let $\chi : [0, \infty) \to [0, 1]$ be a smooth monotonously decreasing function such that $\chi(r) = 1$ for $r \leq \varepsilon/2$ and $\chi(r) = 0$ for $r \geq \varepsilon$.*
Let A be a differential operator of order m on $C^\infty(N, E_N)$.
Then there is $C > 0$ such that for all $x, y \in N$ and for all $t > 0$

$$|A_x k_t(x, y)| \leq C + Ct^{-(n+m)/2} e^{-d(x,y)^2/4t} \chi(d(x,y)) \sum_{i=0}^m d(x,y)^i t^{-\frac{i}{2}} \ .$$

PROOF. This follows from [**BGV**], Prop. 2.46, and its proof. □

PROPOSITION 3.2.2. *Let A be a differential operator of order m on $C^\infty(N, E_N \otimes \mathcal{B})$. Then there is $C > 0$ such that the action of the integral kernel $A_x k_t(x, y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is bounded by $C(1 + t^{-m/2})$ for all $t > 0$.*

PROOF. Choose a finite open covering $\{U_\nu\}_{\nu \in I}$ of N of normal coordinate patches and assume that for every $x, y \in U_\nu$ the shortest geodesic connecting x and y is in U_ν.

Then there are $c_1, c_2 > 0$ such that for all $\nu \in I$ and all $x, y \in U_\nu$

$$c_1|x - y|_\nu \leq d(x, y) \leq c_2|x - y|_\nu \ ,$$

where $| \cdot |_\nu$ denotes the euclidian distance on U_ν defined by the coordinates.

Let $\{\phi_\nu\}_{\nu \in I}$ be a partition of unity subordinate to the covering $\{U_\nu\}_{\nu \in I}$.

Let $\varepsilon > 0$ be smaller than the injectivity radius of N and such that $\{x \in N \mid d(x, \text{supp}\, \phi_\nu) \leq \varepsilon\} \subset U_\nu$ for every $\nu \in I$. Let χ be as in the previous lemma. Then

$$\phi_\nu(x)\chi(d(x,y)) \leq \phi_\nu(x)\chi(c_1|x - y|_\nu)1_{U_\nu}(y) \ .$$

By the previous lemma there is $C > 0$ such that for all $x, y \in N$ and $t > 0$ the term $|A_x k_t(x, y)|$ is bounded by

$$C + Ct^{-(n+m)/2} \sum_{\nu \in J} \phi_\nu(x) \Big(e^{-c_1^2|x-y|_\nu^2/4t} \chi(c_1|x - y|_\nu) \sum_{i=0}^m c_2^i|x - y|_\nu^i t^{-\frac{i}{2}} \Big) 1_{U_\nu}(y) \ .$$

The ν-th term of the outer sum is supported on $U_\nu \times U_\nu$. In the coordinates of U_ν it is of the form $\phi_\nu(x)f_t(x - y)1_{U_\nu}(y)$ with $f_t \in L^1(\mathbb{R}^n)$, and there is $C > 0$ such that

$$\|f_t\|_{L^1} \leq Ct^{n/2}$$

for all $t > 0$.

The assertion follows now from Prop. 5.2.3 and Cor. 5.2.5. □

PROPOSITION 3.2.3.   (1) *The family of integral kernels $k_t(x, y)$ defines a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$, which extends to a bounded holomorphic semigroup. Its generator is $-D_N^2$.*

(2) *The family of integral kernels $k_t(x, y)$ defines a bounded strongly continuous semigroup on $C^m(N, E_N \otimes \mathcal{B})$ for every $m \in \mathbb{N}_0$.*

PROOF. (1) By the previous proposition the action of the integral kernel $k_t(x, y)$ on $L^2(N, E_N \otimes \mathcal{B})$ is uniformly bounded for $t > 0$. On $L^2(N, E_N) \odot \mathcal{B}$ it converges strongly to the identity. Thus $k_t(x, y)$ induces a bounded strongly continuous semigroup on $L^2(N, E_N \otimes \mathcal{B})$.

On $C^\infty(N, E_N \otimes \mathcal{B})$ the action of the generator coincides with the action of $-D_N^2$. Since $C^\infty(N, E_N \otimes \mathcal{B})$ is invariant under the semigroup and dense in $L^2(N, E_N \otimes \mathcal{B})$, it is a core for the generator. Hence the generator is $-D_N^2$.

By the previous proposition there is $C > 0$ such that on $L^2(N, E_N \otimes \mathcal{B})$ for all $0 < t < 1$

$$\|D_N^2 e^{-tD_N^2}\| < Ct^{-1} \ .$$

Since $\operatorname{Ran} e^{-tD_N^2} \subset C^\infty(N, E_N \otimes \mathcal{B}) \subset \operatorname{dom} D_N^2$ for $t > 0$, it follows, by Prop. 5.4.3, that $e^{-tD_N^2}$ extends to a holomorphic semigroup.

The integral kernel of $D_N^2 e^{-tD_N^2}$ is exponentially decaying in the supremum norm for $t \to \infty$, hence $D_N^2 e^{-tD_N^2}$ is exponentially decaying as an operator on $L^2(N, E_N \otimes \mathcal{B})$.

By Prop. 5.4.3 this shows that the holomorphic extension is bounded.

(2) follows from the fact that $k_t(x, y)$ defines a strongly continuous bounded semigroup on $C^m(N, E_N)$ by [**BGV**], Th. 2.30, and that $C^m(N, E_N \otimes \mathcal{B}) \cong C^m(N, E_N) \otimes_\varepsilon \mathcal{B}$.   □

It can be deduced from the asymptotic expansion of the heat kernel ([**BGV**], Th. 2.30) that $e^{-tD_N^2}$ extends even to a holomorphic semigroup on $C^m(N, E_N \otimes \mathcal{B})$ – we do not need this fact in the following.

3.3. The heat semigroup on $[0, 1]$

3.3.1. The semigroup $e^{-tD_I^2}$. Let (P_0, P_1) be a pair of transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$. By Lemma 5.2.10 the operator $I_0 \partial$ with domain $C_R^\infty([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ is closable on $L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$. In order to avoid indices its closure is denoted by D_I (compare with the operator D_I from §2.2.1).

Let $U \in M_{2d}(\mathcal{A}_\infty)$ be a unitary with $UI_0 = I_0 U$ and $UP_0 U^* = P_s$ and let $p \in M_d(\mathcal{A}_\infty)$ be such that

$$UP_1 U^* = \tfrac{1}{2} \begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix} \ .$$

The unitaries U and p exist by Lemma 1.4.3.

PROPOSITION 3.3.1. *Let $\lambda \in \mathbb{C}$ with $\exp(2i\lambda) \notin \sigma(p)$.*

(1) *The operator $D_I - \lambda$ has a bounded inverse on $L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$.*

(2) *The inverse $(D_I - \lambda)^{-1}$ maps $C_R^l([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ isomorphically to $C_R^{l+1}([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ for any $l \in \mathbb{N}_0$.*

(3) *The inverse $(D_I - \lambda)^{-1}$ maps $L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ continuously to $C([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$.*

PROOF. The inverse of $D_I - \lambda$ is given by

$$((D_I - \lambda)^{-1}f)(x) = \int_0^x I_0 e^{-I_0\lambda(x-y)} f(y) \, dy + \int_0^1 e^{-I_0\lambda(x-y)} A(y) f(y) \, dy$$

with

$$A(y) = U^* \frac{i}{e^{2i\lambda} - p} \begin{pmatrix} p & e^{2i\lambda(1-y)} \\ pe^{2i\lambda y} & e^{2i\lambda} \end{pmatrix} U \ .$$

It is straightforward to check that this map fulfills (1), (2) and (3). $\qquad\square$

We define D_I as an unbounded operator on $C_R^l([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$, $l \in \mathbb{N}_0$, by setting $\operatorname{dom} D_I := C_R^{l+1}([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. By the previous proposition D_I is a closed operator on $C_R^l([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

As before D_{I_s} denotes D_I with $R = (P_s, 1 - P_s)$.

We show that $-D_I^2$ generates a holomorphic semigroup on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ and on $C_R^l([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. This will be done by first proving that $-D_{I_s}^2$ generates a holomorphic semigroup and by then applying Prop. 5.4.10.

Moreover we will deduce a norm estimate of the semigroup $e^{-tD_I^2}$ for large t from the knowledge of the resolvent set of $-D_I^2$.

We will use the same method again when we study the heat semigroup on M.

LEMMA 3.3.2. *Assume that* $R = (P_s, 1 - P_s)$.

The operator $-D_{I_s}^2$ *is the generator of a bounded holomorphic semigroup* $e^{-tD_{I_s}^2}$ *on* $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ *and on* $C_R^l([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ *for any* $l \in \mathbb{N}_0$.

PROOF. Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$ (see §3.1.1) and let $S(t)$ be the induced integral operator.

By Lemma 3.1.2 and Prop. 5.2.3 the family $S(t)$ is uniformly bounded on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for $t > 0$ and the family $D_{I_s}^2 S(t)$ is bounded by $C(1+t^{-1})e^{-\omega t}$ for some $C, \omega > 0$ and all $t > 0$.

Since $S(t)$ converges strongly to the identity on $L^2([0,1],\mathbb{C}^{2d}) \odot \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ for $t \to 0$ and has the semigroup property, it is a strongly continuous semigroup on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. By Prop. 5.4.3 it extends to a bounded holomorphic semigroup on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

The kernel H from §3.1.1 defines a bounded holomorphic semigroup on $C^k(\mathbb{R}/4\mathbb{Z},(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well by §5.2.1. This implies that $S(t)$ restricts to a bounded holomorphic semigroup on $C_R^k([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. In particular it follows that $C_R^\infty([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is a core of the generator of $S(t)$. Thus the generator is $-D_{I_s}^2$. $\qquad\square$

PROPOSITION 3.3.3. *The operator* $-D_I^2$ *generates a holomorphic semigroup* $e^{-tD_I^2}$ *on* $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ *as well as on* $C_R^k([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ *for all* $k \in \mathbb{N}_0$, *and there are* $C, \omega > 0$ *such that for all* $t \geq 0$

$$\|e^{-tD_I^2}\| \leq Ce^{-\omega t}$$

on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ *and on* $C_R^k([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

PROOF. The following arguments hold on $L^2([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ as well as on $C_R^k([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$:

The operator $D_I - U^*D_{I_s}U$ is bounded by Prop. 2.2.2. Since $U^*D_{I_s}U$ has a bounded inverse by Prop. 3.3.1 and $-U^*D_{I_s}^2U$ generates a bounded holomorphic

semigroup by the previous lemma, we can apply Prop. 5.4.10. It follows that $-D_I^2$ generates a holomorphic semigroup.

By Prop. 3.3.1 there is $\omega > 0$ such that the spectrum of D_I^2 is a subset of $]\omega, \infty[$. Hence by Prop. 5.4.2 there is $C > 0$ with $\|e^{-tD_I^2}\| \leq Ce^{-\omega t}$.   □

3.3.2. The integral kernel. In this section we prove the existence of the heat kernel for $e^{-tD_I^2}$. By cutting and pasting we construct an approximation of the semigroup $e^{-tD_I^2}$ by integral operators. Using Duhamel's principle (see Prop. 5.4.5) we prove that the error term is an integral operator as well. At the same time we obtain estimates for the integral kernel of $e^{-tD_I^2}$.

The same method will be used later in order to study the heat kernel on M (see §3.5.1) and the heat kernel associated to the superconnection (see §4.3.2).

Let $R = (P_0, P_1)$ be the boundary conditions of D_I.

Let D_{I_0} resp. D_{I_1} be defined as D_I with boundary conditions given by $(P_0, 1 - P_0)$ resp. $(1 - P_1, P_1)$.

From Lemma 1.4.3 and §3.1.1 it follows that $e^{-tD_{I_k}^2}$, $k = 0, 1$, is an integral operator for $t > 0$. Its integral kernel is denoted by $e_t^k(x, y)$.

Let $\phi_0 : [0, 1] \to [0, 1]$ be a smooth function with $\operatorname{supp} \phi_0 \subset [0, \frac{2}{3}[$ and $\operatorname{supp}(1 - \phi_0) \subset]\frac{1}{3}, 1]$ and let $\phi_1 := (1 - \phi_0)$. Furthermore choose smooth functions $\gamma_0, \gamma_1 : [0, 1] \to [0, 1]$ with

- $\gamma_k|_{\operatorname{supp} \phi_k} = 1$, $k = 0, 1$,
- $\operatorname{supp} \gamma_k' \cap \operatorname{supp} \phi_k = \emptyset$, $k = 0, 1$,
- $\operatorname{supp} \gamma_0 \subset [0, \frac{5}{6}]$ and $\operatorname{supp} \gamma_1 \subset [\frac{1}{6}, 1]$.

Write E_t for the integral operator corresponding to the integral kernel

$$e_t(x, y) := \gamma_0(x)e_t^0(x, y)\phi_0(y) + \gamma_1(x)e_t^1(x, y)\phi_1(y) .$$

Set $E_0 := 1$.

Then E_t is strongly continuous on $L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ as well as on $C_R^l([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ at any $t \geq 0$.

For $f \in C_R^\infty([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ the map $[0, \infty) \to L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$, $t \mapsto E_t f$ is even differentiable. Hence by Duhamel's principle (Prop. 5.4.5) we have for $f \in C_R^\infty([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ in $L^2([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$:

$$(*) \qquad e^{-tD_I^2} f - E_t f = -\int_0^t e^{-sD_I^2}(\frac{d}{dt} + D_I^2)E_{t-s} f \ ds .$$

In the following the norm on $M_{2d}(\mathcal{A}_i)$ is denoted by $|\cdot|$.

We define $C_R^k([0, 1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$, $k \in \mathbb{N}_0 \cup \{\infty\}$, as the space of functions in $C^k([0, 1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ whose column vectors are in $C_R^k([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$. Then any bounded operator on $C_R^k([0, 1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ acts as a bounded operator on $C_R^k([0, 1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ in an obvious way.

PROPOSITION 3.3.4. *For $t > 0$ the operator $e^{-tD_I^2}$ is an integral operator. Let k_t be its integral kernel.*

(1) *The map*

$$(0, \infty) \to C^\infty([0, 1], C_R^\infty([0, 1], M_{2d}(\mathcal{A}_\infty))), \ t \mapsto (y \mapsto k_t(\cdot, y))$$

is well-defined and smooth.

(2) $k_t(x,y) = k_t(y,x)^*$.

(3) *For every $m, n \in \mathbb{N}_0$ and every $\delta > 0$ there is $C > 0$ such that*

$$|\partial_x^m \partial_y^n k_t(x,y) - \partial_x^m \partial_y^n e_t(x,y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y,\mathrm{supp}\,\gamma_k')^2}{(4+\delta)t}} 1_{\mathrm{supp}\,\phi_k}(y)$$

for all $t > 0$ and all $x, y \in [0,1]$.

PROOF. Let $f \in C_R^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. From $(*)$ it follows that

$$e^{-tD_I^2}f - E_t f = \sum_{k=0,1} \int_0^t \int_0^1 e^{-sD_I^2}(\gamma_k'\partial + \partial\gamma_k')e_{t-s}^k(\cdot,y)\phi_k(y)f(y)\,dy ds .$$

By Lemma 3.1.1 we can extend the map

$$t \mapsto \left(y \mapsto (\gamma_k'\partial + \partial\gamma_k')e_t^k(\cdot,y)\phi_k(y)\right)$$

by zero to a smooth map from $[0,\infty)$ to $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\mathcal{A}_i)))$.

Since $e^{-sD_I^2}$ acts as a uniformly bounded operator on $C_R^\infty([0,1], M_{2d}(\mathcal{A}_i))$ by Lemma 3.1.1, it follows that the operator on the right hand side is an integral operator with smooth integral kernel.

Hence $e^{-tD_I^2}$ is an integral operator with smooth integral kernel and (1) holds.

The selfadjointness of $e^{-tD_I^2}$ implies (2).

Since $d(\mathrm{supp}\,\phi_k, \mathrm{supp}\,\gamma_k') > \varepsilon$ for some $\varepsilon > 0$, there is $C > 0$, by Lemma 3.1.2, such that for all $x, y \in [0,1]$ and $t > 0$

$$|\partial_x^m \partial_y^n\big(k_t(x,y) - e_t(x,y)\big)|$$

$$\leq C \sum_{k=0,1} \int_0^t \|e^{-sD_I^2}(\gamma_k'\partial + \partial\gamma_k')\partial_y^n\big(e_{t-s}^k(\cdot,y)\phi_k(y)\big)\|_{C^m}\,ds\, 1_{\mathrm{supp}\,\phi_k}(y)$$

$$\leq C \sum_{k=0,1} \int_0^t e^{-\frac{d(y,\mathrm{supp}\,\gamma_k')^2}{(4+\delta)(t-s)}}ds\, 1_{\mathrm{supp}\,\phi_k}(y)$$

$$\leq C \sum_{k=0,1} te^{-\frac{d(y,\mathrm{supp}\,\gamma_k')^2}{(4+\delta)t}}\, 1_{\mathrm{supp}\,\phi_k}(y) .$$

This shows statement (3). □

COROLLARY 3.3.5. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*

For every $m, n \in \mathbb{N}_0$ and $\delta, \varepsilon > 0$ we find $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $t > 0$

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq Ce^{-\frac{d(x,y)^2}{(4+\delta)t}} + Ct \sum_{k=0,1} e^{-\frac{d(y,\mathrm{supp}\,\gamma_k')^2}{(4+\delta)t}} 1_{\mathrm{supp}\,\phi_k}(y) .$$

PROOF. This follows from the previous proposition and Lemma 3.1.2. □

COROLLARY 3.3.6. *Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.*

Let ω be as in Prop. 3.3.3. For every $m, n \in \mathbb{N}_0$ there is $C > 0$ such that for any $t > 0$ and any $x, y \in [0,1]$

$$|\partial_x^m \partial_y^n k_t(x,y)| \leq C(1 + t^{-\frac{m+n+1}{2}})e^{-\omega t} .$$

PROOF. There is $C > 0$ such that for all $x, y \in [0, 1]$ and all $0 < t < 1$

$$|\partial_x^m \partial_y^n k_t(x, y) - \partial_x^m \partial_y^n e_t(x, y)| \leq Ct \sum_{k=0,1} e^{-\frac{d(y, \operatorname{supp} \gamma_k')^2}{5t}} 1_{\operatorname{supp} \phi_k}(y) \ ,$$

hence, by Lemma 3.1.2,

$$|\partial_x^m \partial_y^n k_t(x, y)| \leq C(1 + t^{-\frac{m+n+1}{2}}) \ .$$

For all $t > 1$ and $y \in [0, 1]$

$$k_t(\cdot, y) = e^{-(t-1)D_I^2} k_1(\cdot, y) \ .$$

The assertion follows now since $(y \mapsto k_1(\cdot, y)) \in C^n([0, 1], C_R^m([0, 1], M_{2d}(\mathcal{A}_i)))$ and since by Prop. 3.3.3 the action of $e^{-(t-1)D_I^2}$ on $C_R^m([0, 1], M_{2d}(\mathcal{A}_i))$ is bounded by $Ce^{-\omega t}$ for some $C, \omega > 0$ and every $t > 1$.   □

The following facts will be needed for the definition of the η-form.

LEMMA 3.3.7. *Let k_t be the integral kernel of $e^{-tD_{I_s}^2}$.*
Then for all $x, y \in [0, 1]$ and $t > 0$

$$\operatorname{tr}(D_{I_s})_x k_t(x, y) = 0.$$

PROOF. Let $S := 2P_s - 1 \in \operatorname{Gl}_{2d}(\mathbb{C})$. Then $S^2 = 1$, $SI + IS = 0$, $SP_s = P_s$ and $S(1 - P_s) = -(1 - P_s)$.
This implies $SD_{I_s} e^{-tD_{I_s}^2} + D_{I_s} e^{-tD_{I_s}^2} S = 0$. Therefore

$$S(D_{I_s})_x k_t(x, y) + (D_{I_s})_x k_t(x, y)S = 0 \ ,$$

hence

$$\operatorname{tr}(D_{I_s})_x k_t(x, y) = \operatorname{tr}(-S(D_{I_s})_x k_t(x, y)S) = -\operatorname{tr}(D_{I_s})_x k_t(x, y) \ .$$

It follows that $\operatorname{tr}(D_{I_s})_x k_t(x, y) = 0$.   □

COROLLARY 3.3.8. *Let $(D_I k_t)$ be the integral kernel of $D_I e^{-tD_I^2}$. We have, uniformly on $[0, 1]$:*

$$\lim_{t \to 0} \operatorname{tr}(D_I k_t)(x, x) = 0 \ .$$

PROOF. By the previous lemma $\operatorname{tr}(D_I)_x e_t(x, y) = 0$ for all $x, y \in [0, 1]$. Then the assertion follows from the estimate in Prop. 3.3.4.   □

3.4. The heat semigroup on the cylinder

Let $Z = \mathbb{R} \times [0, 1]$.
Let $R = (P_0, P_1)$ be a pair of pairwise transverse Lagrangian projections of $\mathcal{A}^{2d}$ with $P_0, P_1 \in M_{2d}(\mathcal{A}_\infty)$.
In this section we study the action of D_Z as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$. If not specified the notation is as in §2.3.

First we define the following function spaces and operators:
For $k \in \mathbb{N}_0 \cup \{\infty\}$ let

$$\begin{aligned} &C_R^k(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) \\ &\quad := \ \{f \in C^k(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) \mid (P_i \oplus P_i)(\partial_Z^l f)(x, i) = f(x, i) \\ &\qquad\qquad \text{for } x \in \mathbb{R}; \ i = 0, 1; \ l \in \mathbb{N}_0, \ l \leq k\} \ . \end{aligned}$$

Further suffixes, like c or 0 ..., have their usual meaning.

We endow these spaces with the subspace topologies.

For a Fréchet space V we define the Schwartz space

$$\mathcal{S}(Z, V) := \mathcal{S}(\mathbb{R}) \otimes C^\infty([0,1], V) \ .$$

Moreover let $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ be $\mathcal{S}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \cap C_R^\infty(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as a vector space with the topology induced by $\mathcal{S}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let D_Z as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ be the closure of $\bar{\partial}_Z$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. The existence of the closure follows from Lemma 5.2.10.

Since at the moment it is not clear whether D_Z^2 is closed on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, we define Δ as the closure of $\bar{\partial}_Z^2 = -\partial_{x_1}^2 - \partial_{x_2}^2$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let $\Delta_\mathbb{R}$ be the closure of $-\partial_{x_1}^2$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Let $\tilde{D}_I$ be the closure of $I\partial_{x_2}$ as an unbounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with domain $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

By Prop. 3.3.1 the operator D_I has a bounded inverse on $L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$. By Lemma 5.2.2 the space $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ can be identified with $L^2(\mathbb{R}, L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$, hence $\tilde{D}_I$ has a bounded inverse on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. It follows that $-\tilde{D}_I^2$ is closed.

By an analogous argument $-\tilde{D}_I^2$ generates a bounded holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with integral kernel $k_t^I(x_2, y_2) \oplus k_t^I(x_2, y_2)$ for $t > 0$. Here k_t^I is the integral kernel of $e^{-tD_I^2}$, which exists by Prop. 3.3.4.

Furthermore we have a natural candidate for the integral kernel of a semigroup generated by $-\Delta$, namely

$$k_t^Z(x, y) := \frac{1}{\sqrt{4\pi t}} \, e^{-\frac{(x_1-y_1)^2}{4t}} (k_t^I(x_2, y_2) \oplus k_t^I(x_2, y_2)) \ .$$

In the following let $\omega > 0$ be as in Prop. 3.3.3 such that

$$\|e^{-t\tilde{D}_I^2}\| \leq Ce^{-\omega t}$$

on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ for all $t \geq 0$.

PROPOSITION 3.4.1.   (1) *The integral kernel $k_t^Z(x, y)$ defines a holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ with generator $-\Delta$.*
(2) *For every $m \in \mathbb{N}_0$ the integral kernel $k_t^Z(x, y)$ defines a holomorphic semigroup on $C_R^m(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, denoted by $e^{-t\Delta}$ as well.*
(3) *Let A be a differential operator of order m with coefficients in $C^\infty(Z, M_{4d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$. Then for the operator $Ae^{-t\Delta}$ on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as for $Ae^{-t\Delta} : C_R^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \to C^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, $n \in \mathbb{N}_0$, we have:*
    There is $C > 0$ such that for all $t > 0$

$$\|Ae^{-t\Delta}\| \leq C(1 + t^{-m/2})e^{-\omega t} \ .$$

PROOF. (1) By Prop. 5.2.3 the kernel $\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a uniformly bounded family of operators on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Since it converges strongly to the identity on $L^2(Z)\odot(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}$ for $t\to 0$, it is a strongly continuous semigroup. The space $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the action of the generator on that space is given by $\partial_{x_2}^2$. Hence the generator is $-\Delta_{\mathbb{R}}$.

By checking the assumptions of Prop. 5.4.3 we show that the semigroup $e^{-t\Delta_{\mathbb{R}}}$ extends to a holomorphic one:

The operator $(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}$ equals the convolution with the function

$$g(x_1) := \frac{-i}{\sqrt{4\pi t}}\left(\frac{x_1}{2t}\right)e^{-x_1^2/4t}\ .$$

Since there is $C > 0$ such that for $0 < t < 1$

$$\|g\|_{L^1} \leq Ct^{-1/2}\ ,$$

it follows that

$$\|(i\partial_{x_1})e^{-t\Delta_{\mathbb{R}}}\| \leq Ct^{-1/2}$$

on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, thus for $0 < t < 1$

$$\|\Delta_{\mathbb{R}}e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/2)\Delta_{\mathbb{R}}}\|^2 \leq Ct^{-1}\ .$$

Hence $-\Delta_{\mathbb{R}}$ generates a holomorphic semigroup on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

Note that this estimate also holds on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ showing that $e^{-t\Delta_{\mathbb{R}}}$ is a holomorphic semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well.

Since the semigroups $e^{-t\Delta_{\mathbb{R}}}$ and $e^{-t\tilde{D}_I^2}$ commute with each other, their composition is a holomorphic semigroup. The space $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is invariant under the action of the semigroup and the generator acts on it as $\partial_{x_1}^2 + \partial_{x_2}^2$. Thus the generator is $-\Delta$.

(2) Since for $f \in C_R^n(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, $n \in \mathbb{N}$, we have that $(i\partial_{x_1})e^{-t\Delta}f = e^{-t\Delta}(i\partial_{x_1})f$ and $\tilde{D}_I e^{-t\Delta}f = e^{-t\Delta}\tilde{D}_I f$, the assertion can be reduced to the case $n = 0$.

From Prop. 3.3.3 it follows that the action of the integral kernel $k_t^I(x_2, y_2) \oplus k_t^I(x_2, y_2)$ on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ extends to a holomorphic semigroup. Furthermore in (1) we showed that the integral kernel $\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}$ defines a holomorphic semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence the kernel $k_t^Z(x, y)$ defines a semigroup on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ that extends to a holomorphic one.

(3) We can restrict to the case $n = 0$ by the argument in the proof of (2).

In the following the operator norms can be understood with respect to the action on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ as well as with respect to the action on $C_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

The differential operator A is a sum of operators $a_{hk}\tilde{D}_I^h(i\partial_{x_1})^k$ with $a_{hk} \in C^\infty(Z, M_{4d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i))$ and $h + k \leq m$. We have

$$\tilde{D}_I^h(i\partial_{x_1})^k e^{-t\Delta} = \tilde{D}_I^h e^{-t\tilde{D}_I^2}(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}}\ .$$

By Cor. 5.4.9 there is $C > 0$ such that for $0 < t$

$$\|\tilde{D}_I^h e^{-t\tilde{D}_I^2}\| \leq Ct^{-h/2}e^{-\omega t}\ .$$

By the estimate in the proof of (1)

$$\|(i\partial_{x_1})^k e^{-t\Delta_{\mathbb{R}}}\| \leq \|(i\partial_{x_1})e^{-(t/k)\Delta_{\mathbb{R}}}\|^k \leq Ct^{-k/2}$$

for $0 < t < 1$.

Now the assertion follows by taking into account that $e^{-t\Delta_{\mathbb{R}}}$ is uniformly bounded. □

COROLLARY 3.4.2. *Let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda^2 < \omega$.

(1) *The operator* $D_Z - \lambda$ *is invertible on* $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.
(2) *The operator* $D_Z - \lambda$ *is invertible on* $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.
(3) $\Delta = D_Z^2$.

PROOF. (1) For any seminorm p of $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ there are $C > 0$, $n \in \mathbb{N}$ and a seminorm q such that $p(e^{-t\Delta_{\mathbb{R}}}f) \leq C(1+t^n)q(f)$ and $p(e^{-t\tilde{D}_I^2}f) \leq Ce^{-\omega t}q(f)$ for all $f \in \mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Hence $e^{-t\Delta}$ restricts to a bounded operator on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$, and the integral

$$G(\lambda) = \int_0^\infty (D_Z + \lambda)e^{-t(\Delta - \lambda^2)}f\,dt$$

defines a bounded operator on $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ inverting $D_Z - \lambda$.

(2) The operator $G(\lambda)$ extends to a bounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ since by the previous proposition there is $C > 0$ such that on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$\|(D_Z + \lambda)e^{-t(\Delta - \lambda^2)}\| \leq C(1 + t^{-\frac{1}{2}})e^{-(\omega - \operatorname{Re}\lambda^2)t}$$

for all $t > 0$.

From (1) it follows that $G(\lambda)$ inverts $D_Z - \lambda$ on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

(3) From (2) it follows that the operator D_Z^2 is closed, and from (1) that $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ is a core for D_Z^2. Hence the closure of D_Z^2 equals Δ. □

PROPOSITION 3.4.3. *Let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda < \omega$.

(1) *The operator* $(D_Z^2 - \lambda)^{-1}$ *maps* $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ *continuously to* $C(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.
(2) *Let* $n \in \mathbb{N}, n \geq 2$. *The operator* $(D_Z^2 - \lambda)^{-n}$ *maps* $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ *continuously to* $C^{2n-3}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

PROOF. (1) For $\operatorname{Re}\lambda < \omega$ we have on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$:

$$\begin{aligned}
(D_Z^2 - \lambda)^{-1} &= \int_0^\infty e^{-t(D_Z^2 - \lambda)}dt \\
&= \int_0^\infty e^{\lambda t}e^{-t\Delta_{\mathbb{R}}^2}e^{-t\tilde{D}_I^2}dt\ .
\end{aligned}$$

By Prop. 3.3.1 the operator $D_I^{-1} : L^2([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d}) \to C([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ is bounded. Thus the family of operators

$$e^{-t\tilde{D}_I^2} = \tilde{D}_I^{-1}\tilde{D}_Ie^{-t\tilde{D}_I^2} : L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \to L^2(\mathbb{R}, C([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $C(1 + t^{-\frac{1}{2}})e^{-\omega t}$ for all $t > 0$.

Furthermore the family

$$e^{-t\Delta_{\mathbb{R}}} : L^2(\mathbb{R}, C([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})) \to C(\mathbb{R}, C([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}))$$

is bounded by $\displaystyle\sup_{x_1\in\mathbb{R}} \|\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_1-y_1)^2}{4t}}\|_{L^2_{y_1}}$, hence by $Ct^{-1/4}$ for some $C > 0$.

Thus the integral converges as a bounded operator from $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ to $C(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

(2) We show that for any $k \in \mathbb{N}_0$ with $k \leq 2n - 1$ the map

$$\tilde{D}_I^k(i\partial_{x_1})^{2n-1-k}(D_Z^2 - \lambda)^{-n} : \mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d}) \to L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$$

extends to a bounded operator on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$. Then the assertion follows from the first part.

We have for $k \in \mathbb{N}_0$ with $k \leq 2n + 1$

$$
\begin{aligned}
& \tilde{D}_I^k(i\partial_{x_1})^{2n-k+1}(D_Z^2 - \lambda)^{-n-1} \\
= {} & \int_0^\infty t^{n-1}\tilde{D}_I^k(i\partial_{x_1})^{2n-k+1}e^{-t(D_Z^2-\lambda)}\ dt \\
= {} & \int_0^\infty t^{n-1}\tilde{D}_I^k e^{-t(\tilde{D}_I^2-\lambda)}(i\partial_{x_1})^{2n-k+1}e^{-t\Delta_{\mathbb{R}}}\ dt\ .
\end{aligned}
$$

As bounded operators on $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$\|\tilde{D}_I^k e^{-t(D_I^2-\lambda)}\| \leq Ct^{-k/2}e^{-(\omega-\lambda)t}$$

and

$$\|(i\partial_{x_1})^{2n-k+1}e^{-t\Delta_{\mathbb{R}}}\| \leq C(1 + t^{-(2n-k+1)/2})\ ,$$

hence the integral converges.   □

In the following $|\cdot|$ denotes the norm on $M_{4d}(\mathcal{A}_i)$.

LEMMA 3.4.4. *For every $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{N}_0^2$ there are $c, C > 0$ such that for all $x, y \in Z$ with $d(x, y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^\alpha \partial_y^\beta k_t^Z(x, y)| \leq Ce^{-\frac{d(x,y)^2}{ct}}\ .$$

PROOF. For $m, n \in \mathbb{N}_0$ there are $C, c > 0$ such that

$$|\partial_{x_2}^m \partial_{y_2}^n k_t^I(x_2, y_2)| \leq Ce^{-\frac{(x_2-y_2)^2}{ct}}$$

for $|x_2 - y_2| \geq \varepsilon/2$ and $t > 0$. This follows from Cor. 3.3.5 for $t < 1$ and from Cor. 3.3.6 for $t \geq 1$. For $|x_2 - y_2| \leq \varepsilon/2$ the left hand side is bounded by $C(1 + t^{-\frac{m+n+1}{2}})$ by Cor. 3.3.6. Similar estimates hold for $\frac{1}{\sqrt{4\pi t}}\ e^{-\frac{(x_1-y_1)^2}{4t}}$. A combination of these estimates implies the lemma.   □

In the next lemma $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ is the subspace of $\mathcal{S}(Z, M_{4d}(\mathcal{A}_i))$ of functions whose columns are in $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$. Then operators on $\mathcal{S}_R(Z, \mathcal{A}_i^{4d})$ act on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ columnwise. The space $C_{Rc}^\infty(Z, M_{4d}(\mathcal{A}_i))$ is analogously defined.

LEMMA 3.4.5. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \omega$. Let $\xi_1, \xi_2 \in C^\infty(Z)$ with $\operatorname{supp}\xi_1 \cap \operatorname{supp}\xi_2 = \emptyset$ and assume that $\operatorname{supp}\xi_2$ is compact.*

Then for any $n \in \mathbb{N}$ the operator $\xi_1(D_Z^2 - \lambda)^{-n}\xi_2$ is an integral operator.

Let κ be its integral kernel. Then $(y \mapsto \kappa(\cdot, y)) \in C_c^\infty(Z, \mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i)))$ and $(x \mapsto \kappa(x, \cdot)^*) \in \mathcal{S}(Z, C_{Rc}^\infty(Z, M_{4d}(\mathcal{A}_i)))$.

In particular $\xi_1(D^2 - \lambda)^{-n}\xi_2$ maps $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$ continuously to $\mathcal{S}_R(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

PROOF. First we prove the claim for $n = 1$.

Let $f \in C_{Rc}^{\infty}(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$.

In $L^2(Z, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{4d})$

$$
\begin{aligned}
\xi_1(D_Z^2 - \lambda)^{-1}\xi_2 f &= \int_0^{\infty} \xi_1 e^{-t(D_Z^2 - \lambda)}\xi_2 f \ dt \\
&= \int_0^{\infty}\int_Z \xi_1 k_t^Z(\cdot, y)e^{\lambda t}\xi_2(y)f(y) \ dy dt \ .
\end{aligned}
$$

Let $\varepsilon > 0$ be such that $d(\operatorname{supp}\xi_1, \operatorname{supp}\xi_2) > \varepsilon$.

By the previous lemma there are $c, C > 0$ such that for all $x, y \in Z$ and all $t > 0$

$$
\begin{aligned}
|\xi_1(x)e^{\lambda t}k_t^Z(x,y)\xi_2(y)| &\leq C1_{\{t>1\}}(t)|\xi_1(x)|e^{(\operatorname{Re}\lambda - \omega)t}e^{-\frac{(x_1-y_1)^2}{4t}}|\xi_2(y)| \\
&\quad + C1_{\{t\leq 1\}}(t)|\xi_1(x)|e^{-\frac{d(x,y)^2}{ct}}|\xi_2(y)| \ .
\end{aligned}
$$

Analogous estimates hold for the partial derivatives. Hence we can interchange the order of integration. It follows that $\xi_1(D_Z^2 - \lambda)^{-1}\xi_2$ is an integral operator with integral kernel

$$
\kappa(x, y) := \int_0^{\infty} \xi_1(x)e^{-\lambda t}k_t^Z(x,y)\xi_2(y)dt \ .
$$

For $n = 1$ the other statements of the lemma also follow from the estimates.

For $n > 1$ choose a smooth compactly supported function $\psi : Z \to [0, 1]$ such that $\operatorname{supp}\psi \cap \operatorname{supp}\xi_1 = \emptyset$ and $\operatorname{supp}(1 - \psi) \cap \operatorname{supp}\xi_2 = \emptyset$. Then

$$
\begin{aligned}
\xi_1(D_Z^2 - \lambda)^{-n}\xi_2 &= \xi_1(D_Z^2 - \lambda)^{-1}\psi(D_Z^2 - \lambda)^{-n+1}\xi_2 \\
&\quad + \xi_1(D_Z^2 - \lambda)^{-1}(1 - \psi)(D_Z^2 - \lambda)^{-n+1}\xi_2 \ .
\end{aligned}
$$

By induction the lemma can be applied to $\xi_1(D_Z^2 - \lambda)^{-1}\psi$ and $(1-\psi)(D_Z^2-\lambda)^{-n+1}\xi_2$. The statement of the lemma follows for $\xi_1(D_Z^2 - \lambda)^{-n}\xi_2$ from this and the fact that by Cor. 3.4.2 the operator $(D_Z^2 - \lambda)^{-m}$ acts continuously on $\mathcal{S}_R(Z, M_{4d}(\mathcal{A}_i))$ for all $m \in \mathbb{N}$. $\qquad\square$

3.5. The heat semigroup on M

3.5.1. Definitions. Recall the definition of the operator $D(\rho)^2$ on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$ in §2.1.1 and §2.5. By Lemma 5.2.10 the operator $(\slashed{\partial}_E + \rho K)^2$ with domain $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is closable on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. Its closure will be denoted by $D(\rho)^2$ as well in order to simplify notation.

So far the notation is misleading: It suggests that $D(\rho)^2$ is the square of some unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. This is indeed the case as will become clear in §3.5.5.

Define D_s^2 as a closed operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ in an analogous way.

We will often make use of cutting and pasting arguments on M. We fix the setting:

Let $0 < b_0 \leq \frac{1}{4}$ be small enough and $r_0 > 0$ large enough such that

$$
\operatorname{supp}k_K \cap \big((F(r_0, b_0) \times M) \cup (M \times F(r_0, b_0))\big) = \emptyset \ ,
$$

with $F(r_0, b_0)$ as in §1.1. Here k_K denotes the integral kernel of the operator K from §2.5.

Let $\mathcal{U}(r_0, b_0)$ be the open covering defined in §1.1 and modified in §2.5.

Choose a smooth partition of unity $\{\phi_k\}_{k \in J}$ subordinate to $\mathcal{U}(r_0, b_0)$ and smooth functions $\{\gamma_k\}_{k \in J}$ on M such that for all $k \in J$

- $\operatorname{supp} \gamma_k \subset \mathcal{U}_k$,
- $\operatorname{supp}(1 - \gamma_k) \cap \operatorname{supp} \phi_k = \emptyset$,
- the derivatives $\partial_{e_2}(\phi_k|_F)$ and $\partial_{e_2}(\gamma_k|_F)$ vanish in a neighborhood of ∂M.

Let E_N be a Dirac bundle on a compact spin manifold N that is trivial as a vector bundle and assume that there is a Dirac bundle isomorphism $E|_{\mathcal{U}_\clubsuit} \to E_N$, whose base map is an isometric embedding. Let D_N be the associated Dirac operator. We identify $\mathcal{U}_\clubsuit$ with its image in N and $E|_{\mathcal{U}_\clubsuit}$ with its image in E_N.

Since the support of k_K is in $\mathcal{U}_\clubsuit \times \mathcal{U}_\clubsuit$, the restriction of $D(\rho) = D + \rho K$ to $\mathcal{U}_\clubsuit$ extends to an operator $D_N + \rho K$ on the sections of E_N.

For $k \in \mathbb{Z}/6$ let D_{Z_k} be the operator D_Z on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$ from §3.4 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{(k+1) \bmod 3})$.

3.5.2. The resolvents of $D(\rho)^2$. This section has three different aims:

Using a method of Lott ([**Lo3**], §6.1.) we investigate the resolvent set of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Furthermore we prove a kind of Sobolev embedding theorem – more precisely an analogue of Lemma 3.1.4 for the operator $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Third we obtain more information about the kernel of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$, namely that there is a projection on it and that this projection is a Hilbert-Schmidt operator with a smooth integral kernel.

Let $\omega > 0$ be such that there is $C > 0$ with

$$\|e^{-t D_{Z_k}^2}\| \leq C e^{-\omega t}$$

on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ for all $t \geq 0$ and all $k \in \mathbb{Z}/6$.

Let $\nu \in \mathbb{N}$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < \omega$ we define a parametrix of $(D(\rho)^2 - \lambda)^\nu$:

By Cor. 3.4.2 we can set $Q_k(\lambda) = (D_{Z_k}^2 - \lambda)^{-\nu}$ for $k \in \mathbb{Z}/6$.

Let $Q_\clubsuit(\lambda)$ be a local parametrix of $(D^2 - \lambda)^\nu$ on $\mathcal{U}_\clubsuit$ defined by the symbol of $(D^2 - \lambda)^\nu$ such that $\phi_\clubsuit(Q_\clubsuit(\lambda)(D^2 - \lambda)^\nu - 1)\gamma_\clubsuit$ and $\phi_\clubsuit((D^2 - \lambda)^\nu Q_\clubsuit(\lambda) - 1)\gamma_\clubsuit$ are integral operators with smooth integral kernels.

The operator

$$Q(\lambda) := \sum_{k \in J} \phi_k Q_k(\lambda) \gamma_k$$

acts as a bounded operator on the spaces $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by §5.2.6 and by Cor. 3.4.2.

LEMMA 3.5.1. *For any $\rho \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < \omega$ the operator $Q(\lambda)(D(\rho)^2 - \lambda)^\nu - 1$ restricted to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is an integral operator $\mathcal{K}$ with smooth integral kernel $\kappa \in L^2(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$.*

Furthermore $(x \mapsto \kappa(x, \cdot)^*) \in \mathcal{S}(M, C_{cR}^\infty(M, E \otimes \mathcal{A}_i) \otimes E^*)$ and $(y \mapsto \kappa(\cdot, y)) \in C_c^\infty(M, \mathcal{S}_R(M, E \otimes \mathcal{A}_i) \otimes E^*)$.

In particular $\mathcal{K}$ extends to a bounded operator from $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

PROOF. The difference $(D(\rho)^2 - \lambda)^\nu - (D^2 - \lambda)^\nu$ on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is an integral operator with smooth integral kernel whose support is contained in $\mathrm{supp}\, k_K$. Hence we only need to investigate $Q(\lambda)(D^2 - \lambda)^\nu - 1$.

For any $k \in J$ choose a function $\xi_k \in C_c^\infty(M)$ with values in $[0, 1]$ and such that $\mathrm{supp}\, \xi_k \subset \mathcal{U}_k$, $\xi_k|_{\mathrm{supp}\, d\gamma_k} = 1$ and $\mathrm{supp}\, \phi_k \cap \mathrm{supp}\, \xi_k = \emptyset$. Furthermore assume that $\partial_{e_2}(\xi_k|_F)$ vanishes in a neighborhood of ∂M.

By induction we have that $[\gamma_k, (D^2 - \lambda)^\nu] = \xi_k[\gamma_k, (D^2 - \lambda)^\nu]$ since

$$
\begin{aligned}
&[\gamma_k, (D^2 - \lambda)^\nu] \\
&= (D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) + [\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \\
&= \xi_k(D^2 - \lambda)^{\nu-1}\big(c(d\gamma_k)D + Dc(d\gamma_k)\big) + [\gamma_k, (D^2 - \lambda)^{\nu-1}](D^2 - \lambda) \ .
\end{aligned}
$$

In the following the operators D_{Z_k}, $k \in \mathbb{Z}/6$, are denoted by D as well. Furthermore $\sim$ means equality up to integral operators with smooth compactly supported integral kernels.

Then on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$

$$
\begin{aligned}
Q(\lambda)(D^2 - \lambda)^\nu - 1 &= \sum_{k \in J} \phi_k Q_k(\lambda)[\gamma_k, (D^2 - \lambda)^\nu] + \sum_{k \in J} \phi_k Q_k(\lambda)(D^2 - \lambda)^\nu \gamma_k - 1 \\
&\sim \sum_{k \in J} \phi_k Q_k(\lambda)\xi_k[\gamma_k, (D^2 - \lambda)^\nu] \ .
\end{aligned}
$$

For all $k \in J$ the operator $\phi_k Q_k(\lambda)\xi_k$ is an integral operator whose integral kernel has the properties stated in the lemma. This holds for $k \in \mathbb{Z}/6$ by Lemma 3.4.5 and for $k = \clubsuit$ by the properties of pseudodifferential operators. Now the assertion follows. $\qquad\square$

PROPOSITION 3.5.2. *Let $\rho \in \mathbb{R}$. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < \omega$ such that $D(\rho)^2 - \lambda$ has a bounded inverse on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{A})$.*

Then $D(\rho)^2 - \lambda$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

The inverse $(D(\rho)^2 - \lambda)^{-1}$ acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

PROOF. Let $Q(\lambda)$ and $\mathcal{K}$ be as in the previous lemma such that $Q(\lambda)(D(\rho)^2 - \lambda)f = (1 - \mathcal{K})f$ for $f \in \mathrm{dom}\, D(\rho)^2$.

We want to apply Prop. 5.3.1. Since in general $1 - \mathcal{K}$ is not invertible on $L^2(M, E \otimes \mathcal{A})$, we modify the parametrix:

Choose an integral kernel $s \in C_c^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_i)$ vanishing near $(\partial M \times M) \cup (M \times \partial M)$ such that in $B(L^2(M, E \otimes \mathcal{A}))$

$$\|\mathcal{K} - S(D(\rho)^2 - \lambda)\| \leq \tfrac{1}{2}.$$

This choice is possible since by assumption $(D(\rho)^2 - \lambda)$ has a bounded inverse, hence also $(D(\rho)^2 - \overline{\lambda})$ has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$.

It follows that

$$(Q(\lambda) + S)(D(\rho)^2 - \lambda) = 1 - \big(\mathcal{K} - S(D(\rho)^2 - \lambda)\big)$$

has a bounded inverse on $L^2(M, E \otimes \mathcal{A})$.

Prop. 5.3.1 implies that $1 - \mathcal{K} - S(D(\rho)^2 - \lambda)$ is invertible on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as well. Thus

$$\big(1 - (\mathcal{K} - S(D(\rho)^2 - \lambda))\big)^{-1}(Q(\lambda) + S)$$

is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$, which is a right inverse for $(D(\rho)^2 - \lambda)$. Hence $D(\rho)^2 - \lambda$ is injective and bounded below. It remains to show that its range is dense. This follows from the fact that $D(\rho)^2 - \lambda$ is invertible on $L^2(M, E \otimes \mathcal{A})$.

Since $Q(\lambda)$ acts continuously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and $\mathcal{K}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ continuously to $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by the previous lemma, the operator $(D(\rho)^2 - \lambda)^{-1}$ acts continuously on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by

$$
\begin{aligned}
(D(\rho)^2 - \lambda)^{-1} &= (1 - \mathcal{K})(D(\rho)^2 - \lambda)^{-1} + \mathcal{K}(D(\rho)^2 - \lambda)^{-1} \\
&= Q(\lambda) + \mathcal{K}(D(\rho)^2 - \lambda)^{-1} .
\end{aligned}
$$

$\square$

PROPOSITION 3.5.3. *Let $\rho \in \mathbb{R}$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < \omega$ be such that $(D(\rho)^2 - \lambda)$ has a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.*

Then for $\nu \in \mathbb{N}$, $\nu \geq 2$, the operator $(D(\rho)^2 - \lambda)^{-\nu}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

PROOF. Let $Q(\lambda)(D(\rho)^2 - \lambda)^{\nu} = 1 - \mathcal{K}$ as before, thus

$$
(D(\rho)^2 - \lambda)^{-\nu} = Q(\lambda) + \mathcal{K}(D(\rho)^2 - \lambda)^{-\nu} .
$$

By Prop. 3.4.3 and Lemma 5.2.18 the operator $Q(\lambda)$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ continuously to $C_R^{2\nu-3}(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Furthermore $\mathcal{K}$ is smoothing. $\square$

COROLLARY 3.5.4. *The kernel of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is a subspace of $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.*

PROOF. Let $\lambda \neq 0$ be as in the previous proposition.

Then $(D(\rho)^2 - \lambda)^{-\nu} f = (-\lambda)^{-\nu} f$ for $f \in \operatorname{Ker} D(\rho)^2$ and every $\nu \in \mathbb{N}$. By the previous proposition it follows that the elements of $\operatorname{Ker} D(\rho)^2$ are smooth.

For $k \in \mathbb{Z}/6$ and $f \in \operatorname{Ker} D(\rho)^2$

$$
D_{Z_k}^2 \phi_k f \in C_{cR}^{\infty}(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) .
$$

From Cor. 3.4.2 it follows that

$$
\phi_k f = D_{Z_k}^{-2}(D_{Z_k}^2 \phi_k f) \in \mathcal{S}_R(Z_k, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d}) .
$$

Hence $f \in \mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. $\square$

PROPOSITION 3.5.5. *Let $\rho \neq 0$.*

Let P be the projection onto the kernel of $D(\rho)$ on $L^2(M, E \otimes \mathcal{A})$.

Then P is a finite Hilbert-Schmidt operator whose integral kernel is of the form $\sum_{j=1}^m f_j(x) h_j(y)^*$ with $f_j, h_j \in \operatorname{Ker} D(\rho) \cap \mathcal{S}_R(M, E \otimes \mathcal{A}_{\infty})$.

Furthermore on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ we have that $\operatorname{Ker} D(\rho)^2 = PL^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and $\operatorname{Ran} D(\rho)^2 = (1-P)L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Hence there is a decomposition

$$
L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i) = \operatorname{Ker} D(\rho)^2 \oplus \operatorname{Ran} D(\rho)^2
$$

with respect to which

$$
D(\rho)^2 = 0 \oplus D(\rho)^2|_{\operatorname{Ran} D(\rho)^2} .
$$

Moreover $D(\rho)^2|_{\operatorname{Ran} D(\rho)^2}$ is invertible.

PROOF. First consider the situation on $L^2(M, E \otimes \mathcal{A})$: Since the range of $D(\rho)$ is closed, there is an orthogonal projection P onto the kernel of $D(\rho)$ by Prop. 5.1.12. Furthermore $D(\rho)$ is selfadjoint, hence $\operatorname{Ker} D(\rho) = \operatorname{Ker} D(\rho)^2$. The range of $D(\rho)^2$ is closed, thus zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E \otimes \mathcal{A})$.

Hence, for r small enough,

$$P = \frac{1}{2\pi i} \int_{|\lambda| = r} (D(\rho)^2 - \lambda)^{-1} d\lambda \ .$$

From Prop. 3.5.2 it follows that zero is an isolated point in the spectrum of $D(\rho)^2$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ as well. Thus P is well-defined as a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. By Prop. 5.3.6 it is a Hilbert-Schmidt operator whose integral kernel is as asserted.

The remaining parts follow from the spectral theory for closed operators on Banach spaces ([**Da**], Th. 2.14). $\qquad \square$

COROLLARY 3.5.6. *Let* $\operatorname{Re} \lambda < \omega$.
(1) *Let* $\rho \neq 0$ *and let* P *be the orthogonal projection onto the kernel of* $D(\rho)^2$. *If* $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$, *then* $D(\rho)^2 + P - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *as well.*
(2) *Let* P_0 *be the orthogonal projection onto* $\operatorname{Ker} D_s^2$. *If* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \mathcal{A})$, *then* $D_s^2 + P_0 - \lambda$ *has a bounded inverse on* $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *and the inverse acts as a bounded operator on the space* $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ *as well.*

In particular there is $c > 0$ *such that* $\{\operatorname{Re} \lambda < c\}$ *is in the resolvent set of* $D(\rho)^2 + P$ *resp.* $D_s^2 + P_0$.

PROOF. (1) From the previous proposition it follows that $P(1 - \lambda)^{-1} + (1 - P)(D(\rho)^2 - \lambda)^{-1}$ inverts $D(\rho)^2 + P - \lambda$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Since P acts as a bounded operator on the space $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by Cor. 3.5.4 and $(D(\rho)^2 - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ by Prop. 3.5.2, the operator $(D(\rho)^2 + P - \lambda)^{-1}$ is bounded on $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ as well.

(2) follows analogously. $\qquad \square$

3.5.3. An approximation of the semigroup. By cutting and pasting we construct a family of integral operators that behaves similar to a semigroup generated by $-D(\rho)^2$ for small times.

We work in the setting fixed in §3.5.1.

Let $e(\rho)_t^{\clubsuit}(x, y)$ be the restriction of the integral kernel of $e^{-t(D_N + \rho K)^2}$ to $\mathcal{U}_{\clubsuit} \times \mathcal{U}_{\clubsuit}$ and for $k \in \mathbb{Z}/6$ let $e(\rho)_t^k(x, y)$ be the restriction of the integral kernel of $e^{-t D_{Z_k}^2}$ to $\mathcal{U}_k \times \mathcal{U}_k$. Extend these functions by zero to $M \times M$. Clearly for $k \in \mathbb{Z}/6$ we have $e(\rho)_t^k(x, y) = e(0)_t^k(x, y)$.

We write $E(\rho)_t$ for the family of integral operators on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ corresponding to the integral kernel

$$e(\rho)_t(x, y) := \sum_{k \in J} \gamma_k(x) e(\rho)_t^k(x, y) \phi_k(y)$$

and set $E(\rho)_0 := 1$.

Using the results in §3.2 and §3.4 one deduces:

(1) The family $E(\rho)_t$ is strongly continuous in t on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.
(2) If $f \in C^\infty_{cR}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then the map $[0, \infty) \to L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, $t \mapsto E(\rho)_t f$ is differentiable.
(3) $\operatorname{Ran} E(\rho)_t \subset \mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$.
(4) Let A be a differential operator on $C^\infty(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ of order m with bounded coefficients. For $T > 0$ there is $C > 0$ such that for all $0 < t < T$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$

$$\|AE(\rho)_t\| \leq Ct^{-\frac{m}{2}} \ .$$

(5) For any $m \in \mathbb{N}_0$ and $T > 0$ there is $C > 0$ such that for all $y \in \mathcal{U}_\clubsuit$, $\rho \in [-1, 1]$ and $0 < t < T$ in $C^m(\mathcal{U}_\clubsuit, (E \otimes E_y) \otimes \mathcal{A}_i)$

$$\|e(\rho)^\clubsuit_t(\cdot, y) - e(0)^\clubsuit_t(\cdot, y)\|_{C^m} \leq Ct|\rho| \ .$$

An analogous estimate holds for the partial derivatives in y with respect to unit vector fields on $\mathcal{U}_\clubsuit$.

The last statement follows by Volterra development (Prop. 5.4.4): On N

$$e^{-tD_N(\rho)^2} - e^{-tD_N^2}$$

$$= \rho t \sum_{n=1}^\infty (-1)^n (\rho t)^{n-1} \int_{\Delta^n} e^{-u_0 t D_N^2} \left([D_N, K]_s + \rho K^2\right) e^{-u_1 t D_N^2} \dots$$

$$\dots e^{-u_n t D_N^2} \ du_0 \dots du_n \ ,$$

and the sum is an integral operator whose integral kernel is uniformly bounded in $0 < t < T$ and $\rho \in [-1, 1]$.

3.5.4. The semigroup $e^{-tD_s^2}$. By Cor. 3.1.5 the operator $e^{-tD_s^2}$ on the Hilbert space $L^2(M, E)$ is an integral operator with smooth integral kernel k_t for $t > 0$. In this section we show that k_t defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and that this semigroup extends to a holomorphic one.

For D_s define $e^k_t(x, y)$ analogously to $e(\rho)^k_t(x, y)$ for $D(\rho)$ in the previous section and let

$$e_t(x, y) := \sum_{k \in J} \gamma_k(x) e^k_t(x, y) \phi_k(y) \ .$$

The corresponding family of operators is denoted by E_t. We set $E_0 := 1$. The properties of E_t are as described in the previous section.

For $f \in C^\infty_{cR}(M, E)$

$$e^{-tD_s^2} f - E_t f = -\int_0^t e^{-sD_s^2} \left(\frac{d}{dt} + D_s^2\right) E_{t-s} f \ ds$$

by Duhamel's principle.

PROPOSITION 3.5.7.   (1) *The heat kernel k_t associated to D_s defines a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with generator $-D_s^2$, which extends to a bounded holomorphic semigroup.*
(2) *Let A be a differential operator of order $m \in \mathbb{N}_0$ with bounded smooth coefficients. Then for any $t > 0$ the operator $Ae^{-tD_s^2}$ is bounded on*

$L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ and for any $T > 0$ there is $C > 0$ such that for $0 < t < T$

$$\|A e^{-t D_s^2}\| \leq C t^{-\frac{m}{2}} .$$

PROOF. First we show that for $T > 0$ there is $C > 0$ such that for $0 < t < T$ the difference $A_x k_t(x, y) - A_x e_t(x, y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$.

For $k \in J$ let $\chi_k \in C_c^\infty(M)$ be a function with values in $[0, 1]$, with compact support in $\mathcal{U}_k$ and equal to one on a neighborhood of supp $d\gamma_k$.

From Duhamel's principle it follows that

$$A_x k_t(x, y) - A_x e_t(x, y)$$
$$= -\sum_{k \in J} \int_0^t \int_M A_x k_s(x, r)[(\partial\!\!\!/_E)_r, c(d\gamma_k(r))]_s e_{t-s}^k(r, y) \phi_k(y) \, dr ds .$$

This can be re-written as

$$-\sum_{k \in J} \int_0^t \int_M (1 - \chi_k(x)) A_x k_s(x, r)[(\partial\!\!\!/_E)_r, c(d\gamma_k(r))]_s e_{t-s}^k(r, y) \phi_k(y) \, dr ds$$

$$-\sum_{k \in J} \int_0^t \int_M \chi_k(x) A_x(k_s(x, r) - e_s^k(x, r))[(\partial\!\!\!/_E)_r, c(d\gamma_k(r))]_s e_{t-s}^k(r, y) \phi_k(y) \, dr ds$$

$$-\sum_{k \in J} \int_0^t \int_M \chi_k(x) A_x e_s^k(x, r)[(\partial\!\!\!/_E)_r, c(d\gamma_k(r))]_s e_{t-s}^k(r, y) \phi_k(y) \, dr ds .$$

Using Lemma 3.1.8 and Lemma 3.1.9 we estimate the norms of the three terms in $E_x \otimes E_y$:

Since supp $d\gamma_k \cap$ supp $\phi_k = \emptyset$ and supp $d\gamma_k \cap$ supp$(1 - \chi_k) = \emptyset$, there is $C > 0$ such that for $x, y \in M$ and $t > 0$ the norm of the first term is bounded by

$$C \sum_{k \in J} t(1 - \chi_k(x)) e^{-\frac{d(x, \text{supp } d\gamma_k)^2}{(4+\delta)t}} e^{-\frac{d(y, \text{supp } d\gamma_k)^2}{(4+\delta)t}} 1_{\text{supp } \phi_k}(y) ,$$

and such that for $x, y \in M$ and $t > 0$ the norms of the second and third term are bounded by

$$C t \chi_k(x) e^{-\frac{d(y, \text{supp } d\gamma_k)^2}{(4+\delta)t}} 1_{\text{supp } \phi_k}(y) .$$

When estimating the third term we used that the action of the integral kernel $e_s^k(x, r) \chi_k(r)$ is uniformly bounded for $k = \clubsuit$ on $C^n(\mathcal{U}_\clubsuit, E \otimes E_y)$ by Prop. 3.2.2 and for $k \in \mathbb{Z}/6$ on $C_R^n(\mathcal{U}_k, E \otimes E_y)$ by Prop. 3.4.1.

Analogous estimates hold for the derivatives in y with respect to unit vector fields on M.

Hence $A_x k_t(x, y) - A_x e_t(x, y)$ is bounded by Ct in $L^2(M \times M, E \boxtimes E^*)$ for $0 < t < T$ and some $C > 0$. By Cor. 5.2.4 the corresponding family of operators on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is bounded by Ct for $0 < t < T$, hence $A_x k_t(x, y)$ defines a family of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Write $S(t)$ for the integral operator induced by the integral kernel $k_t(x, y)$.

By property (4) in §3.5.3 there is $C > 0$ such that $\|A E_t\| \leq C t^{-\frac{m}{2}}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ for $0 < t < T$ and some $C > 0$, hence

$$(*) \qquad \|A S(t)\| \leq C t^{-\frac{m}{2}} .$$

The fact that $S(t)$ extends to a bounded holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is an almost immediate consequence of $(*)$:

Since E_t converges strongly to the identity on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ for $t \to 0$, the operator $S(t)$ also does. Furthermore the kernels k_t obey the semigroup law, hence $S(t)$ is a strongly continuous semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

Note that the range of $S(t) - E_t$ is a subset of $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Hence $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is invariant under the action of $S(t)$. It follows that $-D_s^2$ is the generator of $S(t)$.

From Prop. 5.4.3 and the estimate $(*)$ applied to $A = D_s^2$ it follows that the semigroup $S(t) = e^{-tD_s^2}$ extends to a holomorphic semigroup on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

In order to show that the holomorphic semigroup is bounded, let P_0 be the projection onto the kernel of D_s^2.

By Cor. 3.5.6 there is $c > 0$ such that $\{\operatorname{Re} \lambda < c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Hence by Prop. 5.4.2 the holomorphic semigroup $e^{-t(D_s^2 + P_0)}$ is bounded. Thus

$$e^{-tD_s^2} = e^{-t(D_s^2 + P_0)}(1 - P_0) + P_0$$

is bounded as well. $\square$

Recall that in §3.5.1 we fixed the domain of D_s^2 as an unbounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$, but not the domain of D_s. This is done now:

Let D_s be the closure on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ of the Dirac operator $\slashed{\partial}_E$ with domain $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

COROLLARY 3.5.8. *Let P_0 be the projection onto the kernel of D_s^2.*
Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda^2 < 0$.
Then the operators $D_s + P_0$ and $D_s - \lambda$ have a bounded inverse on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

PROOF. By Cor. 3.5.6 there is $c > 0$ such that $\{\operatorname{Re} \lambda \leq c\}$ is in the resolvent set of $D_s^2 + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

By the previous proposition and Prop. 5.4.2 it follows that there is $C > 0$ such that for all $t > 0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$

$$\|(D_s + P_0)e^{-t(D_s^2 + P_0)}\| \leq C t^{-\frac{1}{2}} e^{-ct} .$$

Thus

$$G := \int_0^\infty (D_s + P_0)e^{-t(D_s^2 + P_0)} dt$$

is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$. Furthermore $G = (D_s + P_0)^{-1}$ on the Hilbert space $L^2(M, E)$. From Cor. 3.5.6 it follows that G acts as a bounded operator on $\mathcal{S}_R(M, E)$. Hence G inverts $D_s + P_0$ on $\mathcal{S}_R(M, E) \odot \hat{\Omega}_{\leq \mu} \mathcal{A}_i$, thus G inverts $D_s + P_0$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

The proof of the fact that

$$\int_0^\infty (D_s + \lambda)e^{-t(D_s^2 - \lambda^2)}$$

inverts $(D_s - \lambda)$ for $\operatorname{Re} \lambda^2 < 0$ is analogous. $\square$

This answers a question from the beginning of §3.5.1: the operator D_s^2 is indeed the square of D_s.

3.5.5. The semigroup $e^{-tD(\rho)^2}$. This section is devoted to the study of the holomorphic semigroup generated by $-D(\rho)^2$.

First we prove its existence.

Define $D(\rho)$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ as the closure of the operator $\bar{\partial}_E + \rho K$ with domain $\mathcal{S}_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Let P_0 be the orthogonal projection onto the kernel of D_s^2. Then $D(\rho)$ is a bounded perturbation of $W^*(D_s + P_0)W$ with W as in §2.1.2. By the results of the previous section we can apply Prop. 5.4.10 and conclude that $D(\rho)^2$, as defined in §3.5.1, is the square of $D(\rho)$ and furthermore that $-D(\rho)^2$ generates a holomorphic semigroup. This shows the first assertion of the following proposition.

For $\rho \neq 0$ let P be the orthogonal projection onto the kernel of $D(\rho)^2$.

PROPOSITION 3.5.9. (1) *Let $\rho \in \mathbb{R}$. The operator $-D(\rho)^2$ generates a holomorphic semigroup $e^{-tD(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. For $\rho \neq 0$ the semigroup is bounded holomorphic.*
(2) *For $t \geq 0$ the operator $e^{-tD(\rho)^2}$ depends analytically on ρ. For every $T > 0$ there is $C > 0$ such that*
$$\|e^{-tD(\rho)^2}\| \leq C$$
for all $\rho \in [-1, 1]$ and $0 \leq t \leq T$.
(3) *For $\rho \neq 0$ there are $C, \omega > 0$ such that for all $t \geq 0$*
$$\|(1 - P)e^{-tD(\rho)^2}\| \leq Ce^{-\omega t}.$$

PROOF. (1) Let $\rho \neq 0$. By Cor. 3.5.6 there is $c > 0$ such that $\{\mathrm{Re}\,\lambda \leq c\}$ is in the resolvent set of $D(\rho)^2 + P$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, hence by Lemma 5.4.2 the holomorphic semigroup $e^{-t(D(\rho)^2+P)}$ is bounded by Ce^{-ct} for some $C > 0$, thus for $T > 0$ there is $C > 0$ such that for $t > T$
$$\|D(\rho)^2 e^{-tD(\rho)^2}\| = \|D(\rho)^2 e^{-t(D(\rho)^2+P)}\| \leq Ce^{-ct}.$$

Now Prop. 5.4.3 implies that the semigroup $e^{-tD(\rho)^2}$ is bounded holomorphic.

(2) follows from Prop. 5.4.4.

(3) follows from $(1 - P)e^{-tD(\rho)^2} = (1 - P)e^{-t(D(\rho)^2+P)}$. $\qquad\square$

The following proposition shows that $e^{-tD(\rho)^2}$ is smoothing.

PROPOSITION 3.5.10. (1) *Let $n \in \mathbb{N}_0$. For every $\rho \in \mathbb{R}$ and every $t > 0$ the operator $e^{-tD(\rho)^2}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C_R^n(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.*
(2) *Let $n \in 2\mathbb{N}$, $n \geq 4$. For every $\rho \neq 0$ the family $e^{-tD(\rho)^2} : C_{cR}^n(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C_R^{n-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded.*
(3) *Let $n \in 2\mathbb{N}$, $n \geq 4$. For every $T > 0$ the family $e^{-tD(\rho)^2} : C_{cR}^n(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i) \to C_R^{n-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ is uniformly bounded in $0 \leq t < T$ and in $\rho \in [-1, 1]$.*

PROOF. We conclude (1) from
$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-k}(D(\rho)^2 + 1)^k e^{-tD(\rho)^2},$$
taking into account that $(D(\rho)^2 + 1)^k e^{-tD(\rho)^2}$ is bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $t > 0$ and that $(D(\rho)^2 + 1)^{-k}$ maps $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ continuously to $C_R^{2k-3}(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $k \in \mathbb{N}$, $k \geq 2$, by Prop. 3.5.3.

(2) and (3) follow from

$$e^{-tD(\rho)^2} = (D(\rho)^2 + 1)^{-k} e^{-tD(\rho)^2} (D(\rho)^2 + 1)^k$$

by Prop. 3.5.3.   □

3.5.6. The integral kernel. In order to prove that the operator $e^{-tD(\rho)^2}$ is an integral operator we use the same method as in §3.3.2: Via Duhamel's principle we compare $e^{-tD(\rho)^2}$ and the approximation $E(\rho)_t$, which was defined in §3.5.3.

In the following $|\cdot|$ denotes the norm on the fibers of $(E \boxtimes E^*) \otimes \mathcal{A}_i$.

PROPOSITION 3.5.11. *For every $\rho \in \mathbb{R}$ and every $t > 0$ the operator $e^{-tD(\rho)^2}$ is an integral operator with smooth integral kernel. Let $k(\rho)_t(x,y)$ be its integral kernel.*

(1) *The map $(0,\infty) \to C^\infty(M \times M, (E \boxtimes E^*) \otimes \mathcal{A}_\infty),\ t \mapsto k(\rho)_t$ is smooth.*
(2) $k(\rho)_t(x,y) = k(\rho)_t(y,x)^*$.
(3) *For every $T > 0$ there are $c, C > 0$ such that*

$$|k(\rho)_t(x,y) - e(\rho)_t(x,y)| \leq Ct\big(|\rho| 1_{\mathcal{U}_\clubsuit}(y) + \sum_{k \in J} e^{-\frac{d(y,\mathrm{supp}\, d\gamma_k)^2}{ct}} 1_{\mathrm{supp}\,\phi_k}(y)\big)$$

   for all $0 < t < T$, $\rho \in [-1,1]$ and $x,y \in M$.
(4) *Let $\rho \neq 0$. There are $c, C > 0$ such that*

$$|k(\rho)_t(x,y) - e(\rho)_t(x,y)| \leq Cte^{-\frac{d(y,\mathcal{U}_\clubsuit)^2}{ct}}$$

   for all $t > 0$ and all $x, y \in M$.

Statements analogous to (3) and (4) hold for the partial derivatives in x and y with respect to unit vector fields on M.

PROOF. For (1) it is enough to prove an analogous statement for the operator $e^{-tD(\rho)^2} - E(\rho)_t$.

Let $f \in C^\infty_{cR}(M, E \otimes \mathcal{A}_i)$. Then by Duhamel's principle

$$e^{-tD(\rho)^2} f - E(\rho)_t f$$
$$= -\sum_{k \in J} \int_0^t \int_M e^{-sD(\rho)^2}\big(\frac{d}{dt} + D(\rho)^2\big) \gamma_k e(\rho)^k_{t-s}(\cdot, y) \phi_k(y) f(y)\ dy ds \ .$$

We write

$$\sum_{k \in J} \big(\frac{d}{d\tau} + D(\rho)^2\big) \gamma_k e(\rho)^k_\tau(\cdot, y) \phi_k(y)$$

$$= -\sum_{k \in J} [\bar{\partial}_E, c(d\gamma_k)]_s e(0)^k_\tau(\cdot, y) \phi_k(y)$$

$$- [\bar{\partial}_E, c(d\gamma_\clubsuit)]_s \big(e(\rho)^\clubsuit_\tau(\cdot, y) - e(0)^\clubsuit_\tau(\cdot, y)\big) \phi_\clubsuit(y) \ .$$

Note that the map

$$(0,\infty) \to C^\infty(M, C^\infty_{cR}(M, E \otimes \mathcal{A}_\infty) \otimes E) \ ,$$

$$\tau \mapsto \big(y \mapsto \sum_{k \in J} \big(\frac{d}{d\tau} + D(\rho)^2\big) e(\rho)^k_\tau(\cdot, y) \phi_k(y)\big)$$

is smooth. We show that it extends smoothly by zero to $\tau = 0$. From this and Prop. 3.5.10 it follows that $e^{-tD(\rho)^2} - E(\rho)_t$ is an integral operator.

We estimate the terms on the right hand side of the previous equation.

From the estimates in Lemma 3.1.8 and Lemma 3.4.4 it follows that for any $m \in \mathbb{N}_0$ there are $C, c > 0$ such that

$$\|[\dslash_E, c(d\gamma_k)]_s e(0)^k_\tau(\cdot, y)\phi_k(y)\|_{C^m_R} \leq Ce^{-\frac{d(y, \operatorname{supp} d\gamma_k)^2}{c\tau}} 1_{\operatorname{supp} \phi_k}(y)$$

in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E^*_y$ for all $k \in J$, $0 < \tau < T$, $\rho \in [-1, 1]$ and $y \in M$.

Furthermore by §3.5.3, property (5), there is $C > 0$ such that

$$\|[\dslash_E, c(d\gamma_\clubsuit)]_s \big(e(\rho)^\clubsuit_\tau(\cdot, y) - e(0)^\clubsuit_\tau(\cdot, y)\big)\phi_\clubsuit(y)\|_{C^m_R} \leq C\tau|\rho|1_{\mathcal{U}_\clubsuit}(y)$$

in $C^m_R(M, E \otimes \mathcal{A}_i) \otimes E_y$ for $0 < \tau < T$, $\rho \in [-1, 1]$ and $y \in M$.

Analogous estimates hold for the derivatives in y and also in τ since by the heat equation the derivatives with respect to τ can be expressed in terms of the derivatives with respect to x. This shows (1).

The property $k(\rho)_t(x, y) = k(\rho)_t(y, x)^*$ follows from the selfadjointness of $e^{-tD(\rho)^2}$.

Statement (2) and (3) follow from the estimates using Prop. 3.5.10. For the proof of (3) we also take into account that the kernel $e(\rho)^\clubsuit_t$ and its derivatives are uniformly bounded in t with $t > T$. $\qquad \square$

COROLLARY 3.5.12. *Let $\nu \in \mathbb{N}_0$. For every $\rho \in \mathbb{R}$ and $T > 0$ there is $C > 0$ such that*

$$\|D(\rho)^\nu e^{-tD(\rho)^2}\| \leq Ct^{-\frac{\nu}{2}}$$

for $0 < t < T$ on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$.

PROOF. By Duhamel's principle, for $f \in C^\infty_{Rc}(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$,

$$D(\rho)^\nu e^{-tD(\rho)^2} f - D(\rho)^\nu E(\rho)_t f$$
$$= -\sum_{k \in J} \int_0^t \int_M e^{-sD(\rho)^2} D(\rho)^\nu [D(\rho)^2, \gamma_k]_s e(\rho)^k_{t-s}(\cdot, y)\phi_k(y)f(y) \; dyds \; .$$

There is $C > 0$ such that this term is bounded in $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ by Ct for $0 < t < T$. Furthermore by Prop. 3.2.2 and Prop. 3.4.1 there is $C > 0$ such that on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$

$$\|D(\rho)^\nu E(\rho)_t\| \leq Ct^{-\frac{\nu}{2}} \; .$$

The assertion follows. $\qquad \square$

COROLLARY 3.5.13. *For every $\rho \neq 0$ and $m \in \mathbb{N}$ the family of integral kernels $k(\rho)_t$ defines a strongly continuous semigroup on $C^m_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ bounded by $C(1 + t)^{\frac{3}{2}}$ for some $C > 0$ and all $t > 0$.*

It is denoted by $e^{-tD(\rho)^2}$ as well.

PROOF. By the estimates in the proposition the integral kernel $k(\rho)_t - e(\rho)_t$ defines an operator on $C^m_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ bounded by $Ct^{3/2}$ for every $t > 0$ and some $C > 0$.

We show that $E(\rho)_t$ is a strongly continuous uniformly bounded family of operators on $C^m_R(M, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$.

For $k \in \mathbb{Z}/6$ the action of the integral kernel $e(\rho)^k_t$ on $C^m_R(\mathcal{U}_k, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ is strongly continuous and uniformly bounded in t by Prop. 3.4.1.

By Prop. 3.2.3 and Prop. 5.4.4 the family $e^{-t(D_N+\rho K)^2}$ is a strongly continu-
ous semigroup on $C^m(N, E_N \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$. It is bounded since its integral kernel is
uniformly bounded for $t > 1$. Hence the action of $e(\rho)_t^{\clubsuit}$ on $C^m(\mathcal{U}_{\clubsuit}, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ is
strongly continuous and uniformly bounded.

Since $E(\rho)_t$ converges strongly to the identity on $C_R^m(\mathcal{U}_k, E \otimes \hat{\Omega}_{\leq \mu}\mathcal{A}_i)$ for $t \to 0$,
so does $e^{-tD(\rho)^2}$. It clearly satisfies the semigroup property. □

COROLLARY 3.5.14. *Let $\rho \neq 0$ and $n \in \mathbb{N}$.*
*For every $\varepsilon > 0$ we can find $C, c > 0$ such that for $x, y \in M$ with $d(x, y) > \varepsilon$
and $t > 0$*

$$|k(\rho)_t(x,y)| \leq C\big(e^{-\frac{d(x,y)^2}{ct}} + te^{-\frac{d(y,\mathcal{U}_{\clubsuit})^2}{ct}}\big) .$$

*An analogous statement holds for the derivatives in x and y with respect to unit
vector fields on M.*

We refer to §5.2.4 for the notion of a Hilbert-Schmidt operator and the Hilbert-
Schmidt norm $\| \cdot \|_{HS}$ used in the next corollary.

COROLLARY 3.5.15. *Let $\rho \neq 0$ and $\nu \in \mathbb{N}_0$.*
*The operators $1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}$ and $D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}$ with $r \geq 0$ are Hilbert-
Schmidt operators.*

(1) *For every $T > 0$ there is $C > 0$ such that for any $r \geq 0$ and $t > T$*

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C(1+r)$$

    and

$$\|D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}\|_{HS} \leq C(1+r) .$$

(2) *For every $\varepsilon > 0$ there is $C > 0$ such that for every $r, t > 0$*

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}(1 - 1_{M_{r+\varepsilon}})\|_{HS} \leq C(1+r)t^{1/2}$$

    and

$$\|(1 - 1_{M_{r+\varepsilon}})D(\rho)^\nu e^{-tD(\rho)^2}1_{M_r}\|_{HS} \leq C(1+r)t^{1/2} .$$

PROOF. (1) Since by Prop. 3.5.9 the semigroup $e^{-tD(\rho)^2}$ is bounded, it follows
from Prop. 5.2.13 that there is $C > 0$ such that for all $r > 0$ and $t > T$

$$\|1_{M_r}D(\rho)^\nu e^{-tD(\rho)^2}\|_{HS} \leq C\|1_{M_r}D(\rho)^\nu e^{-TD(\rho)^2}\|_{HS} .$$

By the previous corollary there are $C, c > 0$ such that

$$|1_{M_r}(x)D(\rho)_x^\nu k(\rho)_T(x,y)| \leq C1_{M_r}(x)(e^{-cd(x,y)^2} + e^{-cd(y,M_r)^2})$$

for all $r > 0$ and $x, y \in M$. This yields the asserted estimate. The second estimate
in (1) is proved analogously.

(2) follows from the previous corollary and (1). □

CHAPTER 4

Superconnections and the Index Theorem

The notion of a superconnection on a free finitely generated $\mathbb{Z}/2$-graded module, which we define now, generalizes the notion of a connection on a free module [**Ka**].

In the family case superconnections usually act on infinite dimensional bundles ([**BGV**], Ch. 9). In analogy the superconnections we consider act on modules with infinitely many generators. In that sense the following definition should be merely seen as a motivation for the definitions of the superconnections to come.

DEFINITION 4.0.16. *Let $\mathcal{B}$ be a locally m-convex Fréchet algebra and let $p, q \in \mathbb{N}_0$.*

Let $V := (\mathbb{C}^+)^p \oplus (\mathbb{C}^-)^q$. Consider $V \otimes \hat{\Omega}_* \mathcal{B}$ as a $\mathbb{Z}/2$-graded space.

A SUPERCONNECTION ON $V \otimes \mathcal{B}$ is an odd linear map

$$A : V \otimes \hat{\Omega}_* \mathcal{B} \to V \otimes \hat{\Omega}_* \mathcal{B}$$

satisfying Leibniz rule:

For $\alpha \in V^{\pm} \otimes \hat{\Omega}_k \mathcal{B}$ and $\beta \in \hat{\Omega}_* \mathcal{B}$

$$A(\alpha\beta) = A(\alpha)\beta + (-1)^{\deg \alpha} \alpha \, \mathrm{d} \, \beta$$

where $\deg \alpha$ is the degree of α with respect to the $\mathbb{Z}/2$-grading of $V \otimes \hat{\Omega}_* \mathcal{B}$.

The map A^2 is called the CURVATURE OF A.

As for a connection [**Ka**] the curvature of a superconnection is a right $\hat{\Omega}_* \mathcal{B}$-module map.

4.1. The superconnection A_t^I associated to D_I

4.1.1. The family $e^{-(A_t^I)^2}$. Let C_1 be the $\mathbb{Z}/2$-graded unital algebra generated by an odd element σ with $\sigma^2 = 1$. As a vector space C_1 is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ via the map $C_1 \to \mathbb{C} \oplus \mathbb{C}, a + b\sigma \mapsto (a, b)$. We endow C_1 with the scalar product induced by the standard hermitian scalar product on $\mathbb{C} \oplus \mathbb{C}$.

We identify $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ with $C_1 \otimes L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ and consider D_I, as defined in §3.3.1 with boundary conditions induced by a pair (P_0, P_1), as an unbounded operator on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$.

Let $U \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ be as in Prop. 2.2.1 with $U(0)P_0 U(0)^* = P_s$ and $U(1)P_1 U(1)^* = 1 - P_s$. The map $U^* \, \mathrm{d} \, U$ can be seen as a flat superconnection on $L^2([0,1], C_1 \otimes \mathcal{A}_i^{2d})$. It preserves the space $C_1 \otimes C_R^\infty([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$.

We define

$$A_I := U^* \, \mathrm{d} \, U + \sigma D_I$$

and for $t \geq 0$

$$A_t^I := U^* \, \mathrm{d} \, U + \sqrt{t} \sigma D_I \ .$$

Then A_I is an odd map on $C_1 \otimes C_R^\infty([0,1], (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ fulfilling Leibniz rule and is called a superconnection associated to σD_I. The map A_t^I is called the corresponding rescaled superconnection.

The curvature of A_I is

$$A_I^2 = U^* \, \mathrm{d}^2 \, U + U^* \, \mathrm{d} \, U \sigma D_I + \sigma D_I U^* \, \mathrm{d} \, U + D_I^2 = D_I^2 + \sigma[D_I, U^* \, \mathrm{d} \, U] \ .$$

From

$$
\begin{aligned}
[D_I, U^* \, \mathrm{d} \, U] &= -U^*[\mathrm{d}, U D_I U^*]U \\
&= -U^*([\mathrm{d}, D_{I_s}] + [\mathrm{d}, U I_0(\partial U^*)])U \\
&= -U^* \, \mathrm{d}(U I_0(\partial U^*))U =: R
\end{aligned}
$$

it follows that $A_I^2 = D_I^2 + \sigma R$ with $R \in C^\infty([0,1], M_{2d}(\hat{\Omega}_1 \mathcal{A}_\infty))$ vanishing near the boundary and fulfilling $R^* = -R$.

The curvature of the rescaled superconnection A_t^I is

$$(A_t^I)^2 = t D_I^2 + \sqrt{t}\sigma R \ .$$

We see that the curvature and the rescaled curvature are right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-module maps.

Since A_I^2 is a bounded perturbation of D_I^2, it defines a holomorphic semigroup $e^{-tA_I^2}$ on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

In the following we restrict to $t \geq 0$:

By Volterra development

$$
\begin{aligned}
e^{-tA_I^2} &= \sum_{n=0}^\infty (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D_I^2} \sigma R e^{-u_1 t D_I^2} \sigma R \ldots e^{-u_n t D_I^2} \ du_0 \ldots du_n \\
&= \sum_{n=0}^\infty \sigma^n (-1)^{\frac{(n+1)n}{2}} t^n I_n(t)
\end{aligned}
$$

with

$$I_n(t) := \int_{\Delta^n} e^{-u_0 t D_I^2} R e^{-u_1 t D_I^2} R \ldots e^{-u_n t D_I^2} \ du_0 \ldots du_n \ .$$

Note that the series is finite on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$.

It follows that

$$e^{-(A_t^I)^2} = \sum_{n=0}^\infty \sigma^n (-1)^{\frac{(n+1)n}{2}} t^{n/2} I_n(t) \ .$$

The operators $I_n(t)$ obey the following recursion relation for $n \geq 1$:

$$
\begin{aligned}
I_n(t) &= \int_0^1 du_0 \ e^{-u_0 t D_I^2} R \int_{(1-u_0)\Delta^{n-1}} e^{-u_1 t D_I^2} R \ldots R e^{-u_n t D_I^2} \ du_1 \ldots du_n \\
&= \int_0^1 du_0 \ (1-u_0)^{n-1} e^{-u_0 t D_I^2} R \int_{\Delta^{n-1}} e^{-(1-u_0)u_1 t D_I^2} R \ldots \\
&\qquad \ldots R e^{-(1-u_0)u_n t D_I^2} \ du_1 \ldots du_n \\
&= \int_0^1 du_0 \ (1-u_0)^{n-1} e^{-u_0 t D_I^2} R I_{n-1}((1-u_0)t) \ .
\end{aligned}
$$

Note that $e^{-(A_t^I)^2}$ is selfadjoint on $L^2([0,1], C_1 \otimes (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ in the sense of §5.2.3 and that $I_n(t) = (-1)^n I_n(t)^*$.

4.1.2. The integral kernel of $e^{-(A_t^I)^2}$. Since $e^{-tD_I^2}$ is a bounded semigroup on $C_R^m([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$ for every $m \in \mathbb{N}_0$ by Prop. 3.3.3, the family $I_n(t)$: $C_R^m([0,1],\mathcal{A}_i^{2d}) \to C_R^m([0,1],(\hat{\Omega}_n\mathcal{A}_i)^{2d})$ is uniformly bounded in $t \geq 0$.

In the following we write $|\cdot|$ for the norm on $M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

PROPOSITION 4.1.1. *For every $n \in \mathbb{N}_0$ and $t > 0$ the operator $I_n(t)$ is an integral operator. Let $p_t(x,y)^n$ be its integral kernel.*

(1) *The map*

$$(0,\infty) \to C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_\infty))), \ t \mapsto (y \mapsto p_t(\cdot,y)^n)$$

is smooth.
(2) $p_t(x,y)^n = (-1)^n(p_t(y,x)^n)^*$.
(3) *For every $l,m,n \in \mathbb{N}_0$ there are $C,\omega > 0$ such that for $t > 0$ and $x,y \in [0,1]$*

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t} \ .$$

PROOF. In degree $n = 0$ the assertions hold by Prop. 3.3.4 and Cor. 3.3.6.
Let $k_t(x,y)$ be the integral kernel of $e^{-tD_I^2}$.
For $f \in C_R^\infty([0,1],(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^{2d})$

$$(I_n(t)f)(x) = \int_0^1 \int_0^1 \int_0^1 (1-s)^{n-1} k_{st}(x,r)R(r)p_{(1-s)t}(r,y)^{n-1}f(y) \ dydrds$$

by induction and by the recursion formula above.
We can interchange the integration over r and y.
For the proof of the existence of the integral kernel and of (1) it suffices to show that the map

$$(s,t) \mapsto (y \mapsto \int_0^1 (1-s)^{n-1} k_{st}(\cdot,r)R(r)p_{(1-s)t}(r,y)^{n-1}dr)$$

is a smooth map from $[0,1] \times (0,\infty)$ to $C^\infty([0,1], C_R^\infty[0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i)))$:
For $s \geq \frac{1}{2}$ this follows from the fact that by induction the map

$$(s,t) \mapsto (y \mapsto Rp_{(1-s)t}(\cdot,y)^{n-1})$$

is a smooth map from $[\frac{1}{2},1] \times (0,\infty)$ to $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i)))$. Furthermore the family $e^{-stD_I^2}$ is uniformly bounded on $C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i))$ for any $l \in \mathbb{N}_0$ by Prop. 3.3.3 and depends smoothly on s,t.
For $s \leq \frac{1}{2}$ this holds since

$$(s,t) \mapsto (x \mapsto R^* k_{st}(x,\cdot)^*)$$

is a smooth map with values in $C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i)))$ and since the action of the family $I_{n-1}((1-s)t)^*$ on $C_R^m([0,1], M_{2d}(\hat{\Omega}_{\leq\mu}\mathcal{A}_i)))$ depends smoothly on s,t and is uniformly bounded for any $m \in \mathbb{N}_0$.
Assertion (2) holds since $I_n(t)^* = (-1)^n I_n(t)$.
The preceding arguments and the following facts imply the estimate in (3):
By induction there is $C > 0$ such that the norm of $(y \mapsto Rp_{(1-s)t}(\cdot,y)^{n-1})$ in $C^m([0,1], C_R^l([0,1], M_{2d}(\hat{\Omega}_n\mathcal{A}_i)))$ is bounded by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $0 \leq s \leq \frac{1}{2}$, $t > 0$. Furthermore by Cor. 3.3.6 the norm of $(x \mapsto R^* k_{st}(x,\cdot)^*)$ is bounded in $C^l([0,1], C_R^m([0,1], M_{2d}(\hat{\Omega}_1\mathcal{A}_i))$ by $C(1 + t^{-\frac{l+m+1}{2}})e^{-\omega t}$ for $s > \frac{1}{2}$, $t > 0$. □

The proof of the previous proposition did not use the fact that $D_I^2 + \sigma R$ is the curvature of a superconnection. Hence an analogous argument shows that $e^{-t(D_{I_s}^2 + \sigma R)}$ is an integral operator whose integral kernel can be written as

$$\sum_{n=0}^{\infty} (-1)^{\frac{(n+1)n}{2}} \sigma^n t^n p_t^s(x,y)^n$$

with $\left(y \mapsto p_t^s(\cdot, y)^n\right) \in C^\infty([0,1], C_R^\infty([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i)))$ for $R = (P_s, 1 - P_s)$.

LEMMA 4.1.2. *For every $l, m, n \in \mathbb{N}_0$ and $\varepsilon, \delta > 0$ there is $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^l \partial_y^m p_t^s(x,y)^n| \leq C e^{-\frac{d(x,y)^2}{(4+\delta)t}} .$$

PROOF. In degree $n = 0$ the assertion holds by Lemma 3.1.2.

Let $\eta = \varepsilon/4$.

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function with $\chi(x) = 0$ for $x > \eta$ and $\chi(x) = 1$ for $x \leq \eta/2$.

Let k_t be the integral kernel of $e^{-t D_{I_s}^2}$.

For $l, m \in \mathbb{N}_0$

$$\partial_x^l \partial_y^m p_t^s(x,y)^n$$

$$= \int_0^1 \int_0^1 (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \chi(d(x,r)) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds$$

$$+ \int_0^1 \int_0^1 (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \left(1 - \chi(d(x,r))\right) \ldots$$

$$\ldots \chi(d(r,y)) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds$$

$$+ \int_0^1 \int_0^1 (1-s)^{n-1} \partial_x^l k_{st}(x,r) R(r) \left(1 - \chi(d(x,r))\right) \ldots$$

$$\ldots \left(1 - \chi(d(r,y))\right) \partial_y^m p_{(1-s)t}^s(r,y)^{n-1} dr ds .$$

We begin by estimating the first term on the right hand side: By induction there is $C > 0$ such that for $x, y \in [0,1]$ with $d(x,y) > \varepsilon$, $0 < s < 1$ and $t > 0$

$$\|R\chi(d(\cdot, x)) \partial_y^m p_{(1-s)t}^s(\cdot, y)^{n-1}\|_{C_R^l} \leq C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}$$

in $C_R^l([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$.

Since the operator $e^{-st D_{I_s}^2}$ is uniformly bounded on $C_R^l([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$, the first term is bounded by $C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}}$.

An analogous bound exists for the second term: By Lemma 3.1.2 there is $C > 0$ such that for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and $0 < s < 1$ and $t > 0$ in $C_R^m([0,1], M_{2d}(\hat{\Omega}_n \mathcal{A}_i))$

$$\|\left(\partial_x^l k_{st}(x, \cdot) R\right)^* \left(1 - \chi(d(\cdot, x))\right) \chi(d(\cdot, y))\|_{C_R^m} \leq C e^{-\frac{(d(x,y)-\eta)^2}{(4+\delta)t}} .$$

Furthermore the integral kernel $(y, r) \mapsto (\partial_y^m p_{(1-s)t}^s(r, y)^{n-1})^*$ induces a uniformly bounded family of operators from $C_R^m([0,1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ to $C_R([0,1], M_{2d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$.

The third term is bounded by $C e^{-\frac{d(x,y)^2}{(4+\delta)t}}$ since by Lemma 3.1.2

$$|\partial_x^l k_{st}(x,r) R(r) \left(1 - \chi(d(x,r))\right)| \leq C e^{-\frac{d(x,r)^2}{(4+\delta)st}}$$

and by induction

$$\left|\big(1 - \chi(d(x,r))\big)\partial_y^m p_{(1-s)t}^s(r,y)^{n-1}\right| \leq C e^{-\frac{d(r,y)^2}{(4+\delta)(1-s)t}}$$

for all $x, y, r \in [0,1]$, all $0 < s < 1$ and $t > 0$.

Hence there is $C > 0$ such that for all $x, y \in [0,1]$ and all $t > 0$

$$\left|\partial_x^l \partial_y^m p_t^s(x,y)^n\right| \leq C e^{-\frac{(d(x,y)-\varepsilon/2)^2}{(4+\delta)t}} \ .$$

The assertion follows now from Lemma 3.1.6.

$\square$

As in §3.3.2 we apply Duhamel's principle in order to obtain an estimate for the kernel $p_t(x,y)^n$:

Recall the definitions of $\phi_k, \gamma_k, \ k = 0,1$, in §3.3.2.

By Lemma 1.4.3 we can find unitaries $U_k \in M_{2d}(\mathcal{A}_\infty), \ k = 0,1$, with $U_k I_0 = I_0 U_k$ and $U_k P_k U_k^* = P_s$.

Let $W_n^k(t)$ be the integral operator with integral kernel

$$w_t^k(x,y)^n := U_k^* p_t^s(x,y)^n U_k$$

and denote by $W_n(t)$ the integral operator of

$$w_t(x,y)^n := \gamma_0(x) w_t^0(x,y)^n \phi_0(y) + \gamma_1(x) w_t^1(x,y)^n \phi_1(y) \ .$$

Set $W_0(0) := 1$ and $W_n(0) := 0$ for $n \geq 1$. Then $W_n(t)$ is a strongly continuous family of operators on $L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ for all $n \in \mathbb{N}_0$. For $f \in C_R^\infty([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ the map $[0,\infty) \to L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d}), \ t \mapsto W_n(t)f \in L^2([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$ is even smooth.

Furthermore for $t > 0$ the range of $W_n(t)$ is in $C_R^\infty([0,1], (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{2d})$.

Hence Duhamel's principle yields for $f \in C_R^\infty([0,1], \mathcal{A}_i^{2d})$:

$$\left(e^{-tA_I^2} - \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} t^n W_n(t)\right) f$$

$$= -\int_0^t e^{-sA_I^2}\Big(\frac{d}{dt} + A_I^2\Big) \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} (t-s)^n W_n(t-s)f \ ds$$

$$= \int_0^t e^{-sA_I^2} \sum_{k=0,1} \sum_{n=0}^\infty \sigma^n (-1)^{\frac{n(n+1)}{2}} (t-s)^n [\gamma_k, D_I^2]_s W_n^k(t-s) \phi_k f \ ds$$

$$= \sum_{n=0}^\infty \sigma^n \sum_{j=0}^n (-1)^{k(n,j)} \int_0^t s^{n-j}(t-s)^j I_{n-j}(s) \cdots$$

$$\cdots \sum_{k=0,1} (\gamma_k' \partial + \partial \gamma_k') W_j^k(t-s) \phi_k f \ ds$$

with $k(n,j) = \frac{(n-j)(n-j+1)+j(j+1)}{2}$.

It follows that

$$(*) \qquad \big(I_n(t) - W_n(t)\big)f$$

$$= t^{-n} \sum_{j=0}^n (-1)^{j(n-j)} \int_0^t s^{n-j}(t-s)^j I_{n-j}(s) \sum_{k=0,1} (\gamma_k' \partial + \partial \gamma_k') W_j^k(t-s) \phi_k f \ ds \ .$$

PROPOSITION 4.1.3. *For every $l, m, n \in \mathbb{N}_0$ and every $\delta > 0$ there is $C > 0$ such that*

$$|\partial_x^l \partial_y^m p_t(x,y)^n - \partial_x^l \partial_y^m w_t(x,y)^n| \leq Ct \sum_{k=0,1} e^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $t > 0$ and all $x, y \in [0,1]$.

PROOF. The proof is as for $e^{-tD_I^2}$ (see Prop. 3.3.4).

From the previous lemma it follows that for every $j \in \mathbb{N}_0$ and $k = 0, 1$ there is $C > 0$ such that

$$\|(\gamma_k'\partial + \partial\gamma_k')\partial_y^m\big(w_{t-s}^k(\cdot,y)^j\phi_k(y)\big)\|_{C_R^l} \leq Ce^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $y \in [0,1]$, all $t > 0$ and $0 < s < t$. From this estimate, from the fact that $I_{n-j}(s)$ is a uniformly bounded family of operators on $C_R^l([0,1], M_{2d}(\hat{\Omega}_{\leq n}\mathcal{A}_i))$ and from the equation $(*)$ one deduces as in the proof of Prop. 3.3.4 that

$$\|\partial_y^m p_t(\cdot,y)^n - \partial_y^m w_t(\cdot,y)^n\|_{C_R^l} \leq Ce^{-\frac{d(y,\operatorname{supp}\gamma_k')^2}{(4+\delta)t}} 1_{\operatorname{supp}\phi_k}(y)$$

for all $y \in [0,1]$, all $t > 0$ and $0 < s < t$. $\qquad\square$

COROLLARY 4.1.4. *For every $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that*

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq Ce^{-\frac{d(x,y)^2}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$.

PROOF. Note that the assertion is equivalent to the assertion that for every $l, m, n \in \mathbb{N}_0$ and $\varepsilon > 0$ there are $c, C > 0$ such that

$$|\partial_x^l \partial_y^m p_t(x,y)^n| \leq Ce^{-\frac{1}{ct}}$$

for all $x, y \in [0,1]$ with $d(x,y) > \varepsilon$ and all $t > 0$. The estimate follows for $0 < t < 1$ from the previous proposition and Lemma 4.1.2 since $\operatorname{supp}\gamma_k' \cap \operatorname{supp}\phi_k = \emptyset$. For $t > 1$ it holds by the estimate in Prop. 4.1.1. $\qquad\square$

4.1.3. The η-form. In the following $(D_I p_t)(x,y)^n$ denotes the integral kernel of $D_I I_n(t)$.

LEMMA 4.1.5. *For every $n \in \mathbb{N}_0$ the integral*

$$\int_0^\infty t^{\frac{n-1}{2}} \int_0^1 \operatorname{tr}(D_I p_t)(x,x)^n \, dx \, dt$$

is well-defined in $\hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$.

PROOF. The integral converges in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i / \overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$ for all $n, \mu, i \in \mathbb{N}_0$: For $n = 0$ and $t \to 0$ the convergence follows from Cor. 3.3.8; for $n > 0$ and $t \to 0$ and for $n \in \mathbb{N}_0$ and $t \to \infty$ it follows from Prop. 4.1.1. For the convergence in $\hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s}$ apply Prop. 1.3.5. $\qquad\square$

Let $\operatorname{tr}_\sigma(a + \sigma b) := \operatorname{tr} a$ for $a, b \in M_{2d}(\hat{\Omega}_*\mathcal{A}_i)$ and let Tr_σ be the corresponding trace on integral operators.

DEFINITION 4.1.6. *The η-FORM OF THE SUPERCONNECTION A_I is*

$$\eta(A_I) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \ \in \hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s} \ .$$

Since

$$\eta(A_I) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \int_0^\infty t^{n-\frac{1}{2}} \int_0^1 \mathrm{tr}(D_I p_t)(x,x)^{2n} \ dxdt$$

the η-form is well-defined by the previous lemma.

For $\varepsilon \in]0, \frac{1}{2}]$ let $A_I(\varepsilon) := U_\varepsilon^* \, \mathrm{d} \, U_\varepsilon + \sigma D_I$ be a family of superconnections with $\mathrm{supp}(U_\varepsilon - 1) \subset [0, \varepsilon] \cup [1 - \varepsilon, 1]$.

PROPOSITION 4.1.7. *The limit $\lim_{\varepsilon \to 0} \eta(A_I(\varepsilon))$ exists and does not depend on the choice of U_ε.*

PROOF. Let $R_\varepsilon := [D_I, U_\varepsilon^* \, \mathrm{d} \, U_\varepsilon]$. It suffices to prove that for $y, z \in [0, 1]$ and $s, t > 0$ the limit

$$\lim_{\varepsilon \to 0} \int_0^1 k_s(y, x) R_\varepsilon(x) k_t(x, z) \ dx$$

exists and does not depend on the choice of U_ε.

Let $f(x) := k_s(y, x)$ and $g(x) := k_t(x, z)$.

Then

$$\int_0^1 f(x) R_\varepsilon(x) g(x) \ dx$$

$$= \int_0^1 f(x)[I_0 \partial_x, U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x)] g(x) \ dx$$

$$= \int_0^1 \partial_x \big(f(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g(x) \big) \ dx$$

$$- \int_0^1 f'(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g(x) \ dx - \int_0^1 f(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g'(x) \ dx$$

$$= f(1) I_0 U_\varepsilon^*(1) \, \mathrm{d} \, U_\varepsilon(1) g(1) - f(0) I_0 U_\varepsilon^*(0) \, \mathrm{d} \, U_\varepsilon(0) g(0)$$

$$- \int_0^1 f'(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g(x) \ dx - \int_0^1 f(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g'(x) \ dx$$

$$= - \int_0^1 f'(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g(x) \ dx - \int_0^1 f(x) I_0 U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g'(x) \ dx \ .$$

Since for $\varepsilon \to 0$ and $x \in (0, 1)$ the term $U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g(x)$ resp. $U_\varepsilon^*(x) \, \mathrm{d} \, U_\varepsilon(x) g'(x)$ converges to $\mathrm{d} \, g(x)$ resp. $\mathrm{d} \, g'(x)$, it follows that

$$\lim_{\varepsilon \to 0} \int_0^1 f(x) R_\varepsilon(x) g(x) \ dx = f(0) I_0 \, \mathrm{d} \, g(0) - f(1) I_0 \, \mathrm{d} \, g(1) \ .$$

$\square$

DEFINITION 4.1.8. *We call*

$$\eta(P_0, P_1) := \lim_{\varepsilon \to 0} \eta(A_I(\varepsilon))$$

the η-FORM ASSOCIATED TO THE PAIR (P_0, P_1).

4.2. The superconnection associated to D_Z

Let (P_0, P_1) be a pair of transverse Lagrangian projections with $P_i \in M_{2d}(\mathcal{A}_\infty)$, $i = 1, 2$, and let D_Z be the operator on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$ with boundary conditions (P_0, P_1) as in §3.4.

Let $U \in C^\infty([0, 1], M_{2d}(\mathcal{A}_\infty))$ be as in Prop. 2.2.1 with $U(0)P_0 U(0)^* = P_s$ and $U(1)P_1 U(1)^* = 1 - P_s$. We consider U as a function on Z depending only on the variable x_2 and set

$$\tilde{W} := U \oplus U \in C^\infty(Z, M_{2d}(\mathcal{A}_\infty)) \ .$$

We call $A_Z := \tilde{W}^* \, \mathrm{d}\, \tilde{W} + D_Z$ a superconnection associated to D_Z and $A_t^Z := \tilde{W}^* \, \mathrm{d}\, \tilde{W} + \sqrt{t} D_Z$ the corresponding rescaled superconnection.

The curvature is

$$
\begin{aligned}
A_Z^2 &= D_Z^2 + [\tilde{W}^* \, \mathrm{d}\, \tilde{W}, D_Z]_s \\
&= D_Z^2 + \tilde{W}^*[\mathrm{d}, \tilde{W} c(dx_2) \partial_{x_2} \tilde{W}^*]_s \tilde{W} \\
&= D_Z^2 - c(dx_1) \tilde{W}^*[\mathrm{d}, \tilde{W} I \partial_{x_2} \tilde{W}^*]_s \tilde{W} \\
&= D_Z^2 - c(dx_1) \tilde{W}^* \, \mathrm{d}(\tilde{W} I \partial_{x_2} \tilde{W}^*) \tilde{W} \\
&= D_Z^2 + c(dx_1)(R \oplus (-R))
\end{aligned}
$$

with $R = -U^* \, \mathrm{d}(U I_0 (\partial U^*)) U$.

Let $\tilde{R} = c(dx_1)(R \oplus (-R))$. Then $\tilde{R} \in C^\infty(Z, M_{4d}(\hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ and $\tilde{R}^* = \tilde{R}$.

We have a holomorphic semigroup $e^{-tA_Z^2}$ on $L^2(Z, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^{4d})$.

From §3.4 recall that $e^{-tD_Z^2} = e^{-t\tilde{D}_I^2} e^{-t\Delta_{\mathbb{R}}}$.

By Prop. 4.1.1 the operator

$$I_n(t) = \int_{\Delta^n} e^{-u_0 t D_I^2} R e^{-u_1 t D_I^2} R \ldots e^{-u_n t D_I^2} \ du_0 \ldots du_n$$

is an integral operator. Its integral kernel is denoted by $p_t^I(x, y)^n$.

Since $e^{-t\Delta_{\mathbb{R}}}$ commutes with $\tilde{R}$ and $c(dx_1)$ commutes with $e^{-tD_Z^2}$ and $R \oplus (-R)$, we obtain from Volterra development:

$$
\begin{aligned}
e^{-tA_Z^2} &= \sum_{n=0}^\infty (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D_Z^2} \tilde{R} e^{-u_1 t D_Z^2} \tilde{R} \ldots e^{-u_n t D_Z^2} \ du_0 \ldots du_n \\
&= \sum_{n=0}^\infty (-1)^n t^n e^{-t\Delta_{\mathbb{R}}} \int_{\Delta^n} e^{-u_0 t \tilde{D}_I^2} \tilde{R} e^{-u_1 t \tilde{D}_I^2} \tilde{R} \ldots e^{-u_n t \tilde{D}_I^2} \ du_0 \ldots du_n \\
&= \sum_{n=0}^\infty (-1)^n t^n c(dx_1)^n e^{-t\Delta_{\mathbb{R}}} \left(I_n(t) \oplus (-1)^n I_n(t) \right) \ .
\end{aligned}
$$

We define

$$p_t^Z(x, y)^n := c(dx_1)^n \frac{1}{\sqrt{4\pi t}} \, e^{-\frac{(x_1 - y_1)^2}{4t}} \left(p_t^I(x_2, y_2)^n \oplus (-1)^n p_t^I(x_2, y_2)^n \right) \ .$$

The integral kernel of $e^{-tA_Z^2}$ is

$$\sum_{n=0}^\infty (-1)^n t^n p_t^Z(x, y)^n$$

and the integral kernel of $e^{-(A_t^Z)^2}$

$$\sum_{n=0}^{\infty} (-1)^n t^{\frac{n}{2}} p_t^Z(x,y)^n \ .$$

Note that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^2$

$$\operatorname{tr}_s \partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n = 0 \ .$$

Furthermore

$$(D_Z)_x p_t^Z(x,y)^n = c(dx_1)(\partial_{x_1} + I\partial_{x_2})p_t^Z(x,y)^n \ .$$

Since $\operatorname{tr}_s c(dx_1)\partial_{x_1} p_t^Z(x,y)^n$ vanishes for $x_1 = y_1$ and $\operatorname{tr}_s c(dx_1)I\partial_{x_2} p_t^Z(x,y)^n$ vanishes for all $x,y \in Z$,

$$\operatorname{tr}_s(D_Z)_x p_t^Z(x,y)^n = 0$$

if $x = y$.

Furthermore the integral kernel $p_t^Z(x,y)^n$ satisfies the following Gaussian estimate:

LEMMA 4.2.1. *Let $\alpha, \beta \in \mathbb{N}_0^2$. For every $\varepsilon > 0$ there are $c, C > 0$ such that for all $x,y \in Z$ with $d(x,y) > \varepsilon$ and all $t > 0$*

$$|\partial_x^\alpha \partial_y^\beta p_t^Z(x,y)^n| \leq Ce^{-\frac{d(x,y)^2}{ct}} \ .$$

PROOF. The assertion follows from Prop. 4.1.1 and Cor. 4.1.4. The arguments are as in the proof of Lemma 3.4.4. $\square$

4.3. The superconnection $A(\rho)_t$ associated to $D(\rho)$

4.3.1. The family $e^{-A(\rho)_t^2}$. In §3.5.1 we fixed $r_0, b_0 > 0$ such that

$$\operatorname{supp} k_K \cap \big((F(r_0, b_0) \times M) \cup (M \times F(r_0, b_0))\big) = \emptyset \ .$$

Let $W \in C^\infty(M, \operatorname{End}^+ E \otimes \mathcal{A}_\infty)$ be as in §2.1.2 such that W is parallel on $\{x \in M \mid d(x, \partial M) > b_0\}$. Then $[W, K]_s = 0$.

We define a superconnection associated to $D(\rho)$ on the $\mathbb{Z}/2$-graded $\mathcal{A}_i$-module $L^2(M, E \otimes \mathcal{A}_i)$ by

$$A(\rho) := W^* \, \mathrm{d} \, W + D(\rho) \ .$$

The corresponding rescaled superconnection is

$$A(\rho)_t := W^* \, \mathrm{d} \, W + \sqrt{t}D(\rho) \ .$$

The curvature of $A(\rho)$ is

$$\begin{aligned}
A(\rho)^2 &= W^* \, \mathrm{d}^2 \, W + [D(\rho), W^* \, \mathrm{d} \, W]_s + D(\rho)^2 \\
&= D(\rho)^2 + W^*[\mathrm{d}, Wc(dW^*)]_s W \\
&=: D(\rho)^2 + \mathcal{R} \ .
\end{aligned}$$

We used Prop. 2.1.1 and $[W, K]_s = 0$.

Then $\mathcal{R} \in C^\infty(M, \operatorname{End} E \otimes \hat{\Omega}_{\leq 1}\mathcal{A}_i)$ with $\mathcal{R}^* = \mathcal{R}$. In the flat region

$$\begin{aligned}
\mathcal{R}|_F &= W^*[\mathrm{d}, Wc(e_2)\partial_{e_2}W^*)]_s W \\
&= -c(e_1)W^* \, \mathrm{d}(WI\partial_{e_2}W^*)W \ .
\end{aligned}$$

Furthermore $\mathcal{R}$ vanishes on $\{x \in M \mid d(x, \partial M) > b_0\}$.

For every $k \in \mathbb{Z}/6$ the restriction of $\mathcal{R}$ to $\mathcal{U}_k$ is of the form $c(e_1)(R \oplus (-R))$ with $R \in C^\infty(\mathcal{U}_k, M_{2d}(\hat{\Omega}_{\leq 1}\mathcal{A}_i))$ and R is independent of the variable x_2^k.

The rescaled curvature is

$$A(\rho)_t^2 = tD(\rho)^2 + \sqrt{t}\mathcal{R} \ .$$

As expected, $A(\rho)^2$ and $A(\rho)_t^2$ are right $\hat{\Omega}_{\leq\mu}\mathcal{A}_i$-module homomorphisms.

Since $A(\rho)^2$ is a bounded perturbation of $D(\rho)^2$, it generates a holomorphic semigroup $e^{-tA(\rho)^2}$ on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

In the following we assume that $t \geq 0$.

By Volterra development

$$
\begin{aligned}
e^{-tA(\rho)^2} &= \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{-u_0 tD(\rho)^2} \mathcal{R} e^{-u_1 tD(\rho)^2} \mathcal{R} \dots e^{-u_n tD(\rho)^2} \, du_0 \dots du_n \\
&=: \sum_{n=0}^{\infty} (-1)^n t^n I_n(\rho, t) \ .
\end{aligned}
$$

It follows that

$$e^{-A(\rho)_t^2} = \sum_{n=0}^{\infty} (-1)^n t^{n/2} I_n(\rho, t) \ .$$

For $\rho \neq 0$ the family $t \mapsto I_n(\rho, t)$ is uniformly bounded on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$. By Cor. 3.5.13 it acts as a strongly continuous family of operators on $C_R^m(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and there are $C, l > 0$ such that the action is bounded by $C(1+t)^l$.

4.3.2. The integral kernel of $e^{-A(\rho)_t^2}$. In this section we prove that $I_n(\rho, t)$ is an integral operator for $t > 0$. As usual we construct an approximation of the family $I_n(\rho, t)$ by a family of integral operators and compare it with $I_n(\rho, t)$ by Duhamel's principle.

Let $\mathcal{U}(r_0, b_0) = \{\mathcal{U}_k\}_{k \in J}$, $\{\phi_k\}_{k \in J}$ and $\{\gamma_k\}_{k \in J}$ be as in §3.5.1.

For $k \in \mathbb{Z}/6$ the function $W|_{\mathcal{U}_k} : \mathcal{U}_k \to M_{4d}(\mathcal{A}_\infty)$ does not depend on the coordinate x_1^k. We extend it to a section $\tilde{W}_k : Z_k \to M_{4d}(\mathcal{A}_\infty)$ independent of x_1^k and define the superconnection $A_{Z_k} := \tilde{W}_k^* \, \mathrm{d} \, \tilde{W}_k + D_{Z_k}$, which coincides on $\mathcal{U}_k$ with the superconnection $A(\rho)$.

For $k \in \mathbb{Z}/6$ and $n \in \mathbb{N}_0$ let $w(\rho)_t^k(x, y)^n$ be the restriction of $p_t^{Z_k}(x, y)^n$ to $\mathcal{U}_k \times \mathcal{U}_k$.

Let furthermore $w(\rho)_t^{\clubsuit}(x, y)^0$ be the restriction of the integral kernel of $e^{-tD_N(\rho)^2}$ to $\mathcal{U}_{\clubsuit} \times \mathcal{U}_{\clubsuit}$ and set $w(\rho)_t^{\clubsuit}(x, y)^n = 0$ for $n > 0$.

The reason for this is that $A(\rho)^2$ equals $D(\rho)^2$ on $\mathcal{U}_{\clubsuit}$.

We extend $w(\rho)_t^k(x, y)^n$ by zero to $M \times M$ and set

$$w(\rho)_t(x, y)^n := \sum_{k \in J} \gamma_k(x) w(\rho)_t^k(x, y)^n \phi_k(y) \ .$$

Write $W_n(\rho, t)$ for the corresponding integral operator. It is a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ and on $C_R^m(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$.

Set $W_0(\rho, 0) = 1$ and $W_n(\rho, 0) = 0$ for $n > 0$.

Then for $f \in L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ the family $W_n(\rho, t)f \in L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ depends continuously on t for all $t \in [0, \infty)$, and for $f \in C_{Rc}^\infty(M, E \otimes \mathcal{A}_i)$ even smoothly.

For $f \in C^\infty_{Rc}(M, E \otimes \mathcal{A}_i)$ Duhamel's principle yields:

$$
\left(e^{-tA(\rho)^2} - \sum_{n=0}^\infty (-1)^n t^n W_n(\rho, t) \right) f
$$

$$
= -\int_0^t e^{-sA(\rho)^2} \left(\frac{d}{dt} + A(\rho)^2 \right) \sum_{k \in J} \sum_{n=0}^\infty (-1)^n (t-s)^n \gamma_k W_n^k(\rho, t-s) \phi_k f \ ds
$$

$$
= \int_0^t e^{-sA(\rho)^2} \sum_{k \in J} [\gamma_k, D(\rho)^2]_s \sum_{n=0}^\infty (-1)^n (t-s)^n W_n^k(\rho, t-s) \phi_k f \ ds
$$

$$
= \sum_{n=0}^\infty (-1)^n \int_0^t \sum_{m=0}^n s^{n-m} (t-s)^m I_{n-m}(\rho, s) \cdots
$$

$$
\cdots \sum_{k \in J} [\gamma_k, D(\rho)^2]_s W_m^k(\rho, t-s) \phi_k f \ ds \ .
$$

Hence

$$
\left(I_n(\rho, t) - W_n(\rho, t) \right) f
$$

$$
= -t^{-n} \int_0^t \sum_{m=0}^n s^{n-m} (t-s)^m I_{n-m}(\rho, s) \sum_{k \in J} [\gamma_k, D(\rho)^2]_s W_m^k(\rho, t-s) \phi_k f \ ds \ .
$$

In the following proposition $|\cdot|$ denotes the norm on the fibers of $(E \boxtimes E^*) \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$.

PROPOSITION 4.3.1. *The operator $I_n(\rho, t)$ is an integral operator for $t > 0$. Let $p(\rho)_t(x,y)^n$ be its integral kernel.*

(1) *The map $(0, \infty) \to C^\infty(M \times M, (E \boxtimes E^*) \otimes \hat{\Omega}_n \mathcal{A}_\infty), t \mapsto p(\rho)_t^n$ is smooth.*
(2) $p(\rho)_t(x,y)^n = (p(\rho)_t(y,x)^n)^*$.
(3) *For every $T > 0$ there are $C, c > 0$ such that*

$$
|p(\rho)_t(x,y)^n - w(\rho)_t(x,y)^n| \leq Ct \left(|\rho| 1_{\mathcal{U}_\clubsuit}(y) + \sum_{k \in J} e^{-\frac{d(y, \text{supp } d\gamma_k)^2}{ct}} 1_{\text{supp } \phi_k}(y) \right)
$$

   for all $0 < t < T$, $\rho \in [-1,1]$ and all $x, y \in M$.
(4) *Let $\rho \neq 0$. Then there are $C, c > 0$ and $j \in \mathbb{N}$ such that*

$$
|p(\rho)_t(x,y)^n - w(\rho)_t(x,y)^n| \leq Ct(1+t)^j e^{-\frac{d(y, \mathcal{U}_\clubsuit)^2}{ct}}
$$

   for all $t > 0$ and all $x, y \in M$.

Statements analogous to (3), (4) hold for the partial derivatives of $p(\rho)_t(x,y)^n$ in x of y with respect to unit vector fields on M.

PROOF. The proof follows the proof of Prop. 3.5.11.

In order to show the existence of the integral kernel and (1) we need only investigate $I_n(\rho, t) - W_n(\rho, t)$.

For $f \in C^\infty_{Rc}(M, E \otimes \mathcal{A}_i)$

$$
\left(I_n(\rho, t) - W_n(\rho, t) \right) f
$$

$$
= t^{-n} \int_0^t \int_M \sum_{m=0}^n s^{n-m} (t-s)^m I_{n-m}(\rho, s) \sum_{k \in J} [c(d\gamma_k), D]_s w(\rho)_{t-s}^k(\cdot, y)^m \phi_k(y) f(y) \ dyds \ .
$$

For $\rho \in \mathbb{R}$ and $t > 0$ the family $I_{n-m}(\rho, s)$ is uniformly bounded on $C_R^\nu(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ in $s < t$.

The function

$$\tau \mapsto \big(y \mapsto \sum_{k \in J} [c(d\gamma_k), D]_s w(\rho)_\tau^k(\cdot, y)^m \phi_k(y)\big)$$

is smooth from $(0, \infty)$ to $C^l(M, C_{Rc}^\nu(M, E \otimes \hat{\Omega}_{\leq m+1} \mathcal{A}_i) \otimes E^*)$ for any $l, \nu \in \mathbb{N}_0$.

If $k \in \mathbb{Z}/6$, then by Lemma 4.2.1 there are $c, C > 0$ such that for all $y \in M$ and $0 < \tau$

$$\|[c(d\gamma_k), D]_s w_\tau^k(\cdot, y)^m \phi_k(y)\|_{C^\nu} \leq C e^{-\frac{d(y, \operatorname{supp} d\gamma_k)^2}{c\tau}} 1_{\operatorname{supp} \phi_k}(y) \ .$$

Furthermore $w(\rho)_\tau^\clubsuit(x, y)^m = 0$ for $m > 0$. The kernel $w(\rho)_\tau^\clubsuit(x, y)^0$ is equal to the kernel $e(\rho)_\tau^\clubsuit(x, y)$ in the proof of Prop. 3.5.11 and was estimated there. It follows that for $T, \delta > 0$ there is $C > 0$ such that

$$\|[c(d\gamma_k), D]_s w(\rho)_\tau^\clubsuit(\cdot, y)^0 \phi_\clubsuit(y)\|_{C^\nu} \leq C \big(e^{-\frac{d(y, \operatorname{supp} d\gamma_\clubsuit)^2}{(4+\delta)\tau}} + \tau|\rho|\big) 1_{\operatorname{supp} \phi_\clubsuit}(y)$$

for $y \in M$, $\rho \in [-1, 1]$ and $0 < \tau < T$.

Analogous estimates hold for the derivatives in y and also in τ by the heat equation. This shows that the function above extends smoothly to $\tau = 0$. The existence and (1) and (3) follow now by the usual arguments.

Assertion (2) follows from $I_n(\rho, t)^* = I_n(\rho, t)$.

(4) follows from the estimates by taking into account that for every $\rho \neq 0$ there is $C > 0$ and $j \in \mathbb{N}$ such that the norm of $I_n(\rho, t)$ on $C_R^l(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ is bounded by $C(1 + t)^j$ for all $t > 0$. $\qquad \square$

We deduce the following estimates for later use:

COROLLARY 4.3.2. *Let* $\rho \neq 0$. *For every* $\varepsilon > 0$ *and* $m, n \in \mathbb{N}_0$ *there are* $c, C > 0$ *and* $j \in \mathbb{N}$ *such that for* $t > 0$ *and* $x, y \in M$ *with* $d(x, y) > \varepsilon$

$$|D(\rho)_x^m p(\rho)_t(x, y)^n| \leq C(1 + t)^j \big(e^{-\frac{d(x,y)^2}{ct}} + e^{-\frac{d(y, \mathcal{U}_\clubsuit)^2}{ct}}\big).$$

PROOF. This follows from the proposition and Lemma 4.2.1. $\qquad \square$

COROLLARY 4.3.3. *Let* $k \in \mathbb{Z}/6$.
(1) *For every* $T > 0$ *and* $m, n \in \mathbb{N}_0$ *there are* $c, C > 0$ *such that for all* $x, y \in \mathcal{U}_k$, *for* $0 < t < T$ *and* $\rho \in [-1, 1]$

$$|D(\rho)_x^m p(\rho)_t(x, y)^n - (D_{Z_k})_x^m p_t^{Z_k}(x, y)^n| \leq C e^{-\frac{d(y, \mathcal{U}_\clubsuit)}{ct}} \ .$$

(2) *For every* $\rho \neq 0$ *and* $n \in \mathbb{N}_0$ *there are* $c, C > 0$ *and* $j \in \mathbb{N}$ *such that for all* $x, y \in \mathcal{U}_k$ *and* $t > 0$

$$|D(\rho)_x^m p(\rho)_t(x, y)^n - (D_{Z_k})_x^m p_t^{Z_k}(x, y)^n| \leq C(1 + t)^j e^{-\frac{d(y, \mathcal{U}_\clubsuit)}{ct}} \ .$$

4.4. The index theorem and its proof

4.4.1. The generalized supertrace. In the following tr_s denotes the supertrace on the fibers of $(E \otimes E^*) \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i$ and Tr_s the corresponding supertrace for trace class operators (see §1.3.2).

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function with $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Let $\phi_r : M \to [0, 1]$, $\phi_r(x) := \chi(d(M_r, x))$ for $r > 0$.

DEFINITION 4.4.1. *Let K be a bounded operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ such that $\phi_r K \phi_r$ is a trace class operator for all $r \geq 0$. Then we define the generalized supertrace*

$$\mathrm{Tr}_s K := \lim_{r \to \infty} \mathrm{Tr}_s\big(\phi_r K \phi_r\big)$$

if the limit exists.

For trace class operators the generalized supertrace coincides with the usual one. Note that in general it does not vanish on supercommutators.

PROPOSITION 4.4.2. *For $\rho \in \mathbb{R}$ and $t > 0$ the generalized supertraces*

$$\mathrm{Tr}_s e^{-A(\rho)_t^2}$$

and

$$\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}$$

exist.

PROOF. We show the assertion for $e^{-A(\rho)_t^2}$, the proof for $D(\rho)e^{-A(\rho)_t^2}$ is analogous.

The operator $\phi_r e^{-A(\rho)_t^2} \phi_r$ is trace class for $t > 0$ since

$$\phi_r e^{-A(\rho)_t^2} \phi_r = (\phi_r e^{-A(\rho)_t^2/2})(e^{-A(\rho)_t^2/2} \phi_r)$$

is a product of Hilbert-Schmidt operators. The operators $\phi_r I_n(\rho, t) \phi_r$ are also trace class for $t > 0$.

We show that $\mathrm{tr}_s p(\rho)_t(x, x)^n$ is in $L^1(M, \hat{\Omega}_{\leq \mu} \mathcal{A}_i / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{A}_i, \hat{\Omega}_{\leq \mu} \mathcal{A}_i]_s})$.

By Cor. 4.3.3 there are $c, C > 0$ such that

$$|p(\rho)_t(x, x)^n - p_t^{Z_k}(x, x)^n| \leq Ce^{-cd(x, \mathcal{U}_{\clubsuit})^2}$$

for all $x \in \mathcal{U}_k$.

Since $\mathrm{tr}_s p_t^{Z_k}(x, x)^n = 0$ by §4.2,

$$|\mathrm{tr}_s p(\rho)_t(x, x)^n| \leq Ce^{-cd(x, \mathcal{U}_{\clubsuit})^2} .$$

Now the assertion follows. $\qquad\qquad\square$

4.4.2. The limit of $\mathrm{Tr}_s e^{-A(\rho)_t^2}$ for $t \to \infty$. This section is devoted to the proof of

THEOREM 4.4.3. *Let $\rho \neq 0$.*

Let P_0 be the projection onto the kernel of $D(\rho)$. Then for $T > 0$ there is $C > 0$ such that for all $t > T$

$$|\mathrm{Tr}_s e^{-A(\rho)_t^2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}_s(P_0 W \, \mathrm{d} W^* P_0)^{2n}| \leq Ct^{-\frac{1}{2}}$$

and

$$|\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2}| \leq Ct^{-1}$$

in $\hat{\Omega}_{\leq \mu} \mathcal{A}_i / \overline{[\hat{\Omega}_{\leq \mu} \mathcal{A}_i, \hat{\Omega}_{\leq \mu} \mathcal{A}_i]_s}$.

Note that $(P_0 W \, \mathrm{d} W^* P_0)^{2n} = W(W^* P_0 W)(\mathrm{d}(W^* P_0 W))^{2n} W^*$ by Lemma 5.3.4, hence $(P_0 W \, \mathrm{d} W^* P_0)^{2n}$ is a trace class operator on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$.

The proof is subdivided into some lemmata.

Throughout the section $\rho \neq 0$ is fixed.

In the following $|\cdot|$ denotes the norm on $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/\overline{[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]_s}$ resp. the fiberwise norm of $(E \boxtimes E^*) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ (depending on the context), and $\|\cdot\|$ denotes the operator norm of $B(L^2(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i))$. Furthermore we make use of the Hilbert-Schmidt norm $\|\ \|_{HS}$ and the trace class norm $\|\ \|_{TR}$, which are defined in §5.2.4 and §5.2.5.

LEMMA 4.4.4. *Let $\nu = 0, 1$. For any $T > 0$ there are $\varepsilon, C > 0$ such that for all $t > T$*

$$|\mathrm{Tr}_s D(\rho)^\nu e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t^2 D(\rho)^\nu e^{-A(\rho)_t^2}| \leq Ce^{-\varepsilon t} \ .$$

PROOF. We prove the case $\nu = 0$, the case $\nu = 1$ can be proved analogously.
By Cor. 4.3.3 there are $c, C, r > 0$ such that

$$|p(\rho)_t(x, x)^n - p_t^{Z_k}(x, x)^n| \leq C(1+t)^j e^{-\frac{d(x, M_r)^2}{ct}}$$

for $t > 0$ and $x \in \mathcal{U}_k$ with $k \in \mathbb{Z}/6$.
Hence for $T > 0$ there are $c, C > 0$ such that for all $x \in M$ and $t > T$

$$|\mathrm{tr}_s p(\rho)_t(x, x)^n| \leq Ce^{-\frac{d(x, M_r)^2}{ct}}$$

and thus there are $C, \varepsilon > 0$ such that for all $t > r$

$$\begin{aligned}
|\mathrm{Tr}_s e^{-A(\rho)_t^2} - \mathrm{Tr}_s \phi_t^2 e^{-A(\rho)_t^2}| &= |\mathrm{Tr}_s(1 - \phi_t^2)e^{-A(\rho)_t^2}| \\
&\leq C \int_t^\infty e^{-\frac{(r'-r)^2}{ct}} dr' \\
&\leq Ce^{-\varepsilon t} \ .
\end{aligned}$$

$\square$

Let $P_1 := 1 - P_0$.

LEMMA 4.4.5. *Let $\nu = 0, 1$ and $k \in \mathbb{N}_0$. Then the integral*

$$\int_{\Delta^k} \phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t \ du_0 \ldots du_k$$

converges in the space of trace class operators and for $T > 0$ there are $C, \varepsilon > 0$ such that for all $t > T$ the norm

$$\| \int_{\Delta^k} \left(\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t \right) \ du_0 \ldots du_k \|_{TR}$$

is bounded by $Ce^{-\varepsilon t}$.

PROOF. Note that for any $(u_0, \ldots u_k) \in \Delta_k$ we can find $i \in \{0, 1, \ldots, k\}$ such that $u_i \geq \frac{1}{k+1}$.
We begin by showing that for any $T > 0$ there are $C, \varepsilon > 0$ such that for $\frac{1}{k+1} \leq u_i \leq 1$, for $0 < u_0, \ldots, u_{i-1} \leq 1$ and for $t > T$ the family

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-\frac{u_i}{2} tD(\rho)^2}$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $Cu_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$. If not specified the estimates in the following hold for $\frac{1}{k+1} \leq u_i \leq 1$, for $0 < u_0, \ldots, u_{i-1} \leq 1$ and for $t > T$.

We have that

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-\frac{u_i}{2} t D(\rho)^2}$$

$$= \phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-\frac{u_i}{2} t D(\rho)^2} \phi_{(t+6)}$$

$$+ \phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} \ldots P_1 e^{-u_{i-1} t D(\rho)^2} \mathcal{R} \phi_{(t+6)} P_1 e^{-\frac{u_i}{2} t D(\rho)^2} (1 - \phi_{t+6})$$

$$+ \phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} \ldots P_1 e^{-u_{i-1} t D(\rho)^2} \mathcal{R}(1 - \phi_{(t+6)}) P_1 e^{-\frac{u_i}{2} t D(\rho)^2} (1 - \phi_{(t+6)}) \ .$$

Consider the first term on the right hand side. By Cor. 3.5.15 the Hilbert-Schmidt norm of $e^{-\frac{u_i}{4} t D(\rho)^2} \phi_{(t+6)}$ is bounded by Ct.

Furthermore by Prop. 3.5.9 and Cor. 3.5.12 there are $\varepsilon, C > 0$ such that the operator norm of

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \ldots \mathcal{R} P_1 e^{-\frac{u_i}{4} t D(\rho)^2}$$

is bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$.

Hence (see Prop. 5.2.13) the first term is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$.

In the second term the factor

$$\phi_{(t+6)} P_1 e^{-\frac{u_i}{2} t D(\rho)^2} (1 - \phi_{(t+6)}) = (\phi_{(t+6)} e^{-\frac{u_i}{4} t D(\rho)^2}) P_1 e^{-\frac{u_i}{4} t D(\rho)^2} (1 - \phi_{(t+6)})$$

is a Hilbert-Schmidt operator bounded by $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$. Hence the second term is bounded in the Hilbert-Schmidt norm by $C u_0^{-\frac{\nu}{2}} e^{-\varepsilon t}$.

The estimate of the third term requires more effort. We prove by induction on $j \in \mathbb{N}$ that there is $C > 0$ such that

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} \ldots P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$$

is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by $C u_0^{-\frac{\nu}{2}} (1 + t)$ for $t > T$ and $0 < u_0, \ldots, u_j \le 1$. Then it follows that the third term is uniformly bounded by $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$ since $P_1 e^{-\frac{u_i}{2} t D(\rho)^2}$ is exponentially decaying for $t \to \infty$ by Prop. 3.5.9.

For $j = 0$ the assertion follows from Cor. 3.5.15 by

$$\phi_t D(\rho)^\nu P_1 e^{-u_0 t D(\rho)^2} (1 - \phi_{(t+6)})$$

$$= \phi_t D(\rho)^\nu e^{-u_0 t D(\rho)^2} (1 - \phi_{(t+6)}) - \phi_t D(\rho)^\nu P_0 (1 - \phi_{(t+6)}) \ .$$

Now assume the assertion is true for $j - 1$. We have that

$$\phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} \mathcal{R} P_1 \ldots \mathcal{R} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$$

$$= \phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} P_1 \mathcal{R} P_1 \ldots$$

$$\ldots P_1 e^{-u_{j-1} t D(\rho)^2} P_1 \mathcal{R} \phi_{(t+3)} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$$

$$+ \phi_t P_1 D(\rho)^\nu e^{-u_0 t D(\rho)^2} P_1 \ldots$$

$$\ldots P_1 e^{-u_{j-1} t D(\rho)^2} P_1 (1 - \phi_{(t+3)}) \mathcal{R} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)}) \ .$$

Both terms on the right hand side are bounded in the Hilbert-Schmidt norm by $C u_0^{-\nu/2} (1 + t)$ for all $t > T$ and $0 < u_0, \ldots, u_j \le 1$: the first term since $\phi_{(t+3)} P_1 e^{-u_j t D(\rho)^2} (1 - \phi_{(t+6)})$ is bounded by $C(1 + t)$ by Cor. 3.5.15, the second term by induction.

This shows the claim from the beginning of the proof.

An analogous argument yields that for any $T > 0$ there are $C, \varepsilon > 0$ such that the family

$$P_1 e^{-\frac{u_i}{2}tD(\rho)^2} \mathcal{R} P_1 e^{-u_{i+1}tD(\rho)^2} \mathcal{R} \dots P_1 e^{-u_k tD(\rho)^2} \phi_t$$

is a family of Hilbert-Schmidt operators with Hilbert-Schmidt norm bounded by $Ce^{-\varepsilon t}$ for all $u_i > \frac{1}{k+1}$, $0 \le u_{i+1}, \dots, u_k \le 1$ and $t > T$.

It follows that the integral

$$\int_{\Delta^k} \phi_t P_1 D(\rho)^\nu e^{-u_0 tD(\rho)^2} \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} \dots \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \phi_t \; du_0 \dots du_k$$

converges in the trace class norm and is bounded by $Ce^{-\varepsilon t}$ for some $C, \varepsilon > 0$ and all $t > T$. $\qquad \square$

We have that

$$\phi_t D(\rho)^\nu e^{-A(\rho)_t^2} \phi_t = \sum_{k=0}^{\infty} (-1)^k t^{k/2} \; \phi_t D(\rho)^\nu I_k(\rho, t) \phi_t \; .$$

The decomposition $e^{-u_i tD(\rho)^2} = P_0 + P_1 e^{-u_i tD(\rho)^2}$ induces a decomposition of $D(\rho)^\nu I_k(\rho, t)$ into a sum of 2^{k+1} terms. For $0 \le j \le k+1$ let $P_{jk}^\nu(t)$ be the sum of those terms with exactly j factors of the form $P_1 e^{-u_i tD(\rho)^2}$.

Thus

$$D(\rho)^\nu e^{-A(\rho)_t^2} = \sum_{k=0}^{\infty} (-1)^k t^{\frac{k}{2}} \sum_{j=0}^{k+1} P_{jk}^\nu(t) \; .$$

Note that Prop. 3.5.5 implies that P_0 is a trace class operator, hence the operators $P_{jk}^\nu(t)$ with $j \ne k+1$ are trace class for $t \ne 0$.

From $P_0 \mathcal{R} P_0 = P_0[W^* \, dW, D(\rho)]_s P_0 = 0$ it follows that $P_{jk}^\nu(t) = 0$ for $j < \frac{k}{2}$.

Furthermore for k even

$$\begin{aligned}
P_{\frac{k}{2}k}^\nu(t) &= \int_{\Delta^k} D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_3 tD(\rho)^2} \mathcal{R} P_0 \dots \\
&\qquad \dots P_0 \mathcal{R} P_1 e^{-u_{k-1} tD(\rho)^2} \mathcal{R} P_0 \; du_0 \dots du_k \; .
\end{aligned}$$

Since $D(\rho)P_0 = 0$, it follows that $P_{\frac{k}{2}k}^1(t) = 0$.

Moreover by the previous lemma

$$\|\phi_t P_{(k+1)k}^\nu(t) \phi_t\|_{TR} \le Ce^{-\varepsilon t}$$

for t large.

Now we study the behavior of the remaining cases for large t.

LEMMA 4.4.6. *Let $\nu = 0, 1$ and $j, k \in \mathbb{N}_0$ with $\frac{k}{2} \le j \le k$. For every $T > 0$ there is $C > 0$ such that for $t > T$*

$$\|P_{jk}^\nu(t)\|_{TR} \le Ct^{-j} \; .$$

If k is odd, then for every $n \in \mathbb{N}$ and $T > 0$ there is $C > 0$ such that for $t > T$

$$|\mathrm{Tr}_s \phi_t^2 P_{\frac{k+1}{2}k}^\nu(t)| \le Ct^{-n} \; .$$

PROOF. For $j \le k$ the operator $P_{jk}^\nu(t)$ is a sum of terms of the form

$$\int_{\Delta^k} (A(u_0, \dots u_i, t)P_0)(P_0 B(u_{i+1}, \dots u_k, t)) \; du_0 \dots du_k \; ,$$

where A and B are continuous families of bounded operators on $L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i)$ for $u_0 \neq 0$.

Since P_0 is a Hilbert-Schmidt operator by Prop. 3.5.5, Prop. 5.2.13 implies that

$$\| \int_{\Delta^k} A(u_0, \ldots u_i, t) P_0 \big) \big(P_0 B(u_{i+1}, \ldots u_k, t) \, du_0 \ldots du_k \|_{TR}$$

$$\leq C \|P_0\|_{HS}^2 \int_{\Delta^k} \|A(u_0, \ldots u_i, t)\| \|B(u_{i+1}, \ldots u_k, t)\| \, du_0 \ldots du_k \ .$$

Let $\omega > 0$ be such that there is $C > 0$ with $\|P_1 e^{-tD(\rho)^2}\| \leq C e^{-\omega t}$ for all $t > 0$. Then

$$\|P_{jk}^\nu(t)\|_{TR}$$

$$\leq \ C \int_0^1 du_0 \ (u_0 t)^{-\nu/2} e^{-\omega u_0 t} \int_{(1-u_0)\Delta^{k-1}} \exp(-\sum_{i=1}^{j-1} \omega u_i t) \, du_1 \ldots du_k$$

$$= \ C \int_0^1 du_0 \ (u_0 t)^{-\nu/2} e^{-\omega u_0 t} \int_0^{1-u_0} e^{-\omega st} \operatorname{vol}(s\Delta^{j-2}) \operatorname{vol}\big((1-u_0-s)\Delta^{k-j}\big) \, ds$$

$$= \ C \int_0^1 du_0 \ (u_0 t)^{-\nu/2} e^{-\omega u_0 t} \int_0^{1-u_0} e^{-\omega st} \frac{s^{j-2}(1-u_0-s)^{k-j}}{(j-2)!(k-j)!} ds$$

$$= \ C t^{-j} \int_0^t dy \ y^{-\nu/2} e^{-\omega y} \int_0^{t-y} e^{-\omega x} \frac{x^{j-2}(1-y/t-x/t)^{k-j}}{(j-2)!(k-j)!} \, dx$$

$$\leq \ C t^{-j} \Big(\int_0^\infty y^{-\nu/2} e^{-\omega y} \, dy \Big) \Big(\int_0^\infty e^{-\omega x} \frac{x^{j-2}}{(j-2)!(k-j)!} \, dx \Big)$$

$$\leq \ C t^{-j} \ .$$

This shows the first statement.

For k odd

$$\operatorname{Tr}_s \phi_t^2 P_{\frac{k+1}{2} k}^\nu(t)$$

$$= \ \int_{\Delta^k} \operatorname{Tr}_s \phi_t^2 D(\rho)^\nu P_0 \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} P_0 \ldots P_0 \mathcal{R} P_1 e^{-u_k tD(\rho)^2} \, du_0 \ldots du_k$$

$$+ \ \int_{\Delta^k} \operatorname{Tr}_s D(\rho)^\nu P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 \ldots e^{-u_{k-1} tD(\rho)^2} \mathcal{R} P_0 \phi_t^2 \, du_0 \ldots du_k \ .$$

By Prop. 3.5.5 we can estimate $\|(1 - \phi_t^2)P_0\|_{HS}$ and $\|P_0(1 - \phi_t^2)\|_{HS}$ by Ct^{-n}. The second estimate follows then from the cyclicity of the supertrace since $P_0 P_1 = P_1 P_0 = 0$. $\qquad\square$

From the estimates so far obtained the second assertion of the theorem follows.
Furthermore the previous lemmas imply that for $t > T$ there is $C > 0$ such that

$$|\operatorname{Tr}_s \phi_t^2 e^{-A(\rho)_t^2} - \sum_{n=0}^\infty t^n \ \operatorname{Tr}_s \phi_t^2 P_{n(2n)}^0(t)| \leq C t^{-\frac{1}{2}} \ .$$

Hence it remains to study the behavior of $t^n P_{n(2n)}(t)$ for $t \to \infty$.

Recall that

$$P_{n(2n)}^0(t) \ = \ \int_{\Delta^{2n}} P_0 \mathcal{R} P_1 e^{-u_0 tD(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 tD(\rho)^2} \mathcal{R} P_0 \ldots$$

$$\ldots P_0 \mathcal{R} P_1 e^{-u_{n-1} tD(\rho)^2} \mathcal{R} P_0 \, du_0 \ldots du_{2n} \ .$$

By Prop. 3.5.5 for any $j \in \mathbb{N}$ there is $C > 0$ such that

$$\|P_0 - \phi_t^2 P_0\|_{HS} \leq Ct^{-j} \ ,$$

hence

$$|\mathrm{Tr}_s \phi_t^2 P_{n(2n)}^0(t) - \mathrm{Tr}_s P_{n(2n)}^0(t))| \leq Ct^{-j} \ .$$

In the next lemma we show that for $T > 0$ there is $C > 0$ such that

$$\|t^n P_{n(2n)}^0(t) - (-1)^n \frac{1}{n!} (P_0 W dW^* P_0)^{2n}\| \leq Ct^{-1}$$

in $B(L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ for $t > T$. This implies that there is $C > 0$ such that for $t > T$

$$|t^n \mathrm{Tr}_s P_{n(2n)}^0(t) - (-1)^n \frac{1}{n!} \mathrm{Tr}_s (P_0 W dW^* P_0)^{2n}|$$

$$\leq \quad C \|P_0\|_{HS}^2 \|t^n P_{n(2n)}^0(t) - (-1)^n \frac{1}{n!} (P_0 W dW^* P_0)^{2n}\| \leq Ct^{-1} \ .$$

Both estimates combined yield the first assertion of the theorem.

LEMMA 4.4.7. *Let* $k, n \in \mathbb{N}_0$ *with* $n \leq k$. *For* $t \to \infty$ *the term*

$$t^n \int_{\Delta^k} P_0 \mathcal{R} P_1 e^{-u_0 t D(\rho)^2} P_1 \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \dots$$

$$\dots P_0 \mathcal{R} P_1 e^{-u_{n-1} t D(\rho)^2} \mathcal{R} P_0 \ du_0 \dots du_k$$

converges in $B(L^2(M, E \otimes \hat{\Omega}_{\leq \mu} \mathcal{A}_i))$ *to*

$$(-1)^n \frac{1}{(k-n)!} \ (P_0 W \, \mathrm{d} W^* P_0)^{2n}$$

with $O(t^{-1})$.

PROOF. For $i, j \in \{0, 1\}$ write $_i d_j := P_i W \, \mathrm{d} W^* P_j$.
By $\mathcal{R} = [W^* \, \mathrm{d} W, D(\rho)]_s$

$$P_1 \mathcal{R} P_0 \quad = \quad P_1 D(\rho) W^* \, \mathrm{d} W P_0 \ ,$$
$$P_0 \mathcal{R} P_1 \quad = \quad P_0 W^* \, \mathrm{d} W D(\rho) P_1 \ ,$$

thus

$$P_0 \mathcal{R} P_1 e^{-t D(\rho)^2} \mathcal{R} P_0 = {}_0 d_1 D(\rho)^2 e^{-t D(\rho)^2} {}_1 d_0 \ .$$

This term is uniformly bounded for $t \to 0$. Hence the integral

$$\int_0^t {}_0 d_1 D(\rho)^2 e^{-s D(\rho)^2} {}_1 d_0 \ ds$$

converges and equals $_0 d_1 (e^{-t D(\rho)^2} - 1) \ _1 d_0$.
For $n \in \mathbb{N}$ and $k \geq n$

$$t^n \int_{\Delta^k} P_0 \mathcal{R} P_1 e^{-u_0 t D(\rho)^2} \mathcal{R} P_0 \mathcal{R} P_1 e^{-u_1 t D(\rho)^2} \mathcal{R} P_0 \dots$$

$$\dots P_0 \mathcal{R} P_1 e^{-u_{n-1} t D(\rho)^2} \mathcal{R} P_0 \ du_0 \dots du_k$$

$$= t^n \int_{\Delta^k} {}_0 d_1 D(\rho)^2 e^{-u_0 t D(\rho)^2} {}_1 d_0 {}_0 d_1 D(\rho)^2 e^{-u_1 t D(\rho)^2} {}_1 d_0 \dots$$

$$\dots {}_0 d_1 D(\rho)^2 e^{-u_{n-1} t D(\rho)^2} {}_1 d_0 \ du_0 \dots du_k \ .$$

Set

$$D_n := \{(u_0, u_1, \ldots u_{n-1}) \in \mathbb{R}^n \mid \sum_{i=0}^{n-1} u_i \leq 1; \ 0 \leq u_i \leq 1, \ i = 0, \ldots, n-1\} \ .$$

By integration on $u_n, \ldots, u_k$ the previous term equals

$$(*) \qquad t^n \int_{D_n} \ldots {}_0d_1 D(\rho)^2 e^{-u_{n-1}tD(\rho)^2} {}_1d_0 \ \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i)\Delta^{k-n}) \ du_0 \ldots du_{n-1} \ .$$

We claim that $(*)$ converges to $\frac{1}{(k-n)!}({}_0d_{11}d_0)^n$ with $O(t^{-1})$. Then it follows from $(W^* \, \mathrm{d} \, W)^2 = 0$ that ${}_0d_{11}d_0 = -{}_0d_0^2$. This shows the assertion of the lemma.

For $n = 1$ and $k = 1$ the term $(*)$ equals 1.

For $n = 1$ and $k > 1$ the claim follows by induction since by partial integration

$$\int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \, \mathrm{vol}((1 - u_0)\Delta^{k-1}) \ du_0$$

$$= \quad {}_0d_1 e^{-tD(\rho)^2} {}_1d_0 + \int_0^1 {}_0d_1 D(\rho)^2 e^{-u_0 tD(\rho)^2} {}_1d_0 \, \mathrm{vol}((1 - u_0)\Delta^{k-2}) \ du_0$$

because of

$$\partial_{u_0} \mathrm{vol}((1 - u_0)\Delta^{k-1}) = - \, \mathrm{vol}((1 - u_0)\Delta^{k-2}) \ .$$

Furthermore ${}_0d_1 e^{-tD(\rho)^2} {}_1d_0$ decays exponentially for $t \to \infty$.

For general k and n the term $(*)$ equals, by partial integration on u_{n-1},

$$t^n \int_{\Delta^{n-1}} \ldots {}_0d_1 \left[-t^{-1}e^{-xtD(\rho)^2} \mathrm{vol}((1 - \sum_{i=0}^{n-2} u_i - x)\Delta^{k-n}) \right]_0^{u_{n-1}} {}_1d_0 \ du_0 \ldots du_{n-1}$$

$$+ \quad t^{n-1} \int_{D_n} \ldots {}_0d_1 e^{-u_{n-1}tD(\rho)^2} {}_1d_0 \ \partial_{u_{n-1}} \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i)\Delta^{k-n}) \ du_0 \ldots du_{n-1} \ .$$

Note that the first integral vanishes for $x = u_{n-1}$.

We obtain

$$t^{n-1} \int_{D_{n-1}} \ldots {}_0d_1 D(\rho)^2 e^{-u_{n-2}tD(\rho)^2} {}_1d_{00}d_{11}d_0 \, \mathrm{vol}((1 - \sum_{i=0}^{n-2} u_i)\Delta^{k-n}) \ du_0 \ldots du_{n-2}$$

$$-t^{n-1} \int_{D_{n-1}} \ldots {}_0d_1 e^{-u_{n-1}tD(\rho)^2} {}_1d_0 \, \mathrm{vol}((1 - \sum_{i=0}^{n-1} u_i)\Delta^{k-n-1}) \ du_0 \ldots du_{n-1} \ .$$

There are $C, \omega > 0$ such that the last term is bounded by

$$Ct^{n-1} \int_0^1 e^{-s\omega t} \, \mathrm{vol}(s\Delta^{n-1}) \, \mathrm{vol}((1-s)\Delta^{k-n-1}) \ ds \ ,$$

hence it vanishes with $O(t^{-1})$ for $t \to \infty$ (by a calculation as the proof of the previous lemma).

By induction the first term converges with $O(t^{-1})$ to

$$\frac{1}{(k - 1 - (n - 1))!}({}_0d_{11}d_0)^n = \frac{1}{(k - n)!}({}_0d_{11}d_0)^n \ .$$

$\square$

4.4.3. The limit of $\mathrm{Tr}_s e^{-A_t^2}$ for $t \to 0$. Recall the definition of $\mathcal{N}$ from §2.5.

THEOREM 4.4.8.   (1) $\displaystyle\lim_{t \to 0} \mathrm{Tr}_s e^{-A_t^2} = \mathcal{N}$.

(2) $\displaystyle\lim_{t \to 0} \mathrm{Tr}_s D e^{-A_t^2} = 0$.

PROOF. (1) By Prop. 4.3.1

$$\lim_{t \to 0} \mathrm{Tr}_s e^{-A_t^2} = \sum_{n=0}^{\infty} (-1)^n \lim_{t \to 0} \ t^{\frac{n}{2}} \sum_{k \in J} \int_{\mathcal{U}_k} \gamma_k(x) \ \mathrm{tr}_s w(0)_t^k (x,x)^n \phi_k(x) \ dx \ .$$

Now $\mathrm{tr}_s w(\rho)_t^k (x,x)^n = 0$ for $k \in \mathbb{Z}/6$ and for all $n \in \mathbb{N}_0$ and $t > 0$ by §4.2.

Furthermore $w(0)_t^{\clubsuit}(x,y)^n = 0$ for $n > 0$.

Recall that $\mathcal{U}_{\clubsuit}$ contains the isolated point $*$. Since $w(0)_t^{\clubsuit}(x,y)^0$ is the integral kernel of $e^{-tD_N^2}$,

$$\lim_{t \to 0} \mathrm{tr}_s w(0)_t^{\clubsuit}(x,x)^0 = \mathcal{N} 1_*(x)$$

in $C(\mathcal{U}_{\clubsuit})$ by the local index theorem ([**BGV**], Th. 4.2) and by $\mathrm{ch}(E/S) = \mathrm{ch}((\mathbb{C}^+)^d) + \mathrm{ch}((\mathbb{C}^-)^d) = 0$.

(2) Prop. 4.3.1 implies that

$$\lim_{t \to 0} \mathrm{Tr}_s D e^{-A_t^2} = \sum_{n=0}^{\infty} (-1)^n \lim_{t \to 0} \ t^{\frac{n}{2}} \ \mathrm{Tr}_s D W_n(0,t) \ .$$

Since $D W_0(0,t)$ is an odd trace class operator, its supertrace vanishes. Furthermore $\mathrm{Tr}_s D W_n(0,t) = 0$ by §4.2.   □

4.4.4. $\dfrac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2}$ and $\dfrac{d}{d\rho} \mathrm{Tr}_s e^{-A(\rho)_t^2}$.

LEMMA 4.4.9.   (1)

$$\frac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s \frac{dA(\rho)_t^2}{dt} e^{-A(\rho)_t^2} \ .$$

(2)

$$\frac{d}{d\rho} \mathrm{Tr}_s e^{-A(\rho)_t^2} = -\,\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

PROOF. (1) First we calculate $\frac{d}{dt} e^{-A(\rho)_t^2}$.

Consider the holomorphic semigroup $e^{-t'(D(\rho)^2 + z\mathcal{R})}$ depending on the parameter z. By the semigroup law

$$\frac{d}{dt'} \ e^{-t'(D(\rho)^2 + z\mathcal{R})} = -(D(\rho)^2 + z\mathcal{R}) e^{-t'(D(\rho)^2 + z\mathcal{R})}$$

and by Duhamel's formula (Prop. 5.4.4)

$$\frac{d}{dz} \ e^{-t'(D(\rho)^2 + z\mathcal{R})} = -\int_0^{t'} e^{-(t'-s)(D(\rho)^2 + z\mathcal{R})} \mathcal{R} e^{-s(D(\rho)^2 + z\mathcal{R})} ds \ .$$

It follows that

$$
\begin{aligned}
\frac{d}{dt} \, e^{-A(\rho)_t^2} &= \frac{d}{dt} \, e^{-t(D(\rho)^2 + t^{-1/2}\mathcal{R})} \\
&= \frac{d}{dt'} \, e^{-t'(D(\rho)^2 + t^{-1/2}\mathcal{R})}\big|_{(t'=t)} - \frac{1}{2}t^{-3/2}\frac{d}{dz} \, e^{-t(D(\rho)^2 + z\mathcal{R})}\big|_{(z=t^{-1/2})} \\
&= -(D(\rho)^2 + t^{-1/2}\mathcal{R})e^{-t(D(\rho)^2 + t^{-1/2}\mathcal{R})} \\
&\quad + \frac{1}{2}t^{-3/2}\int_0^t e^{-(t-s)(D(\rho)^2 + t^{-1/2}\mathcal{R})}\mathcal{R}e^{-s(D(\rho)^2 + t^{-1/2}\mathcal{R})}\,ds \\
&= -t^{-1}A(\rho)_t^2 e^{-A(\rho)_t^2} + \frac{1}{2}\,t^{-1/2}\int_0^1 e^{-(1-s)A(\rho)_t^2}\mathcal{R}e^{-sA(\rho)_t^2}\,ds \ .
\end{aligned}
$$

Using this equation we prove that

$$
(*_1) \qquad |\frac{d}{dt}\,\mathrm{tr}_s p(\rho)_t(x,x)^n| \leq Ce^{-cd(x,M_r)^2}
$$

for all x uniformly in t for $t \in [t_1, t_2]$ with $t_1, t_2 > 0$. This yields that

$$
\frac{d}{dt}\,\mathrm{Tr}_s e^{-A(\rho)_t^2} = \mathrm{Tr}_s \frac{d}{dt} e^{-A(\rho)_t^2} \ .
$$

By Cor. 4.3.3 and the fact that the pointwise supertrace of the integral kernel $(A_t^{Z_k})^2 e^{-(A_t^{Z_k})^2}$ vanishes for $k \in \mathbb{Z}/6$ the pointwise supertrace of the integral kernel of $A(\rho)_t^2 e^{-A(\rho)_t^2}$ can be estimated by $Ce^{-cd(x,M_r)^2}$ uniformly in t on compact subsets of $(0,\infty)$.

The integral kernel of $\int_0^1 e^{-(1-s)A(\rho)_t^2}\mathcal{R}e^{-sA(\rho)_t^2}\,ds$ is the sum over $m,n \in \mathbb{N}_0$ of the terms

$$
(-1)^{m+n}t^{\frac{m+n}{2}}\int_0^1 (1-s)^{m/2}s^{n/2}\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \ dyds \ .
$$

We decompose

$$
\begin{aligned}
&p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \\
&= p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)\big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big) \\
&\quad + \big(p(\rho)_{(1-s)t}(x,y)^m - w(\rho)_{(1-s)t}(x,y)^m\big)\mathcal{R}(y)w(\rho)_{st}(y,x)^n \\
&\quad + w(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)w(\rho)_{st}(y,x)^n \ .
\end{aligned}
$$

By Prop. 4.3.1 and the fact that the operator $I_m(\rho, (1-s)t)$ is uniformly bounded on $C_R(M, E \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ for $0 \leq (1-s)t \leq t_2$ there are $C, c, r > 0$ such that for all $x \in M$ and $0 \leq s \leq 1$ and $t_1 \leq t \leq t_2$

$$
|\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)\big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n\big)dy| \leq Ce^{-cd(x,M_r)^2} \ .
$$

An analogous estimate holds for the second term.

By §4.2 and since $\mathcal{R}|_{\mathcal{U}_\clubsuit} = 0$

$$
\mathrm{tr}_s w(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)w(\rho)_{st}(y,x)^n = 0 \ .
$$

We conclude that there are $r, c, C > 0$ such that for $x \in M_r$, $0 \leq s \leq 1$ and $t_1 \leq t \leq t_2$

$$
(*_2) \qquad |\mathrm{tr}_s\int_M p(\rho)_{(1-s)t}(x,y)^m\mathcal{R}(y)p(\rho)_{st}(y,x)^n \ dy| \leq Ce^{-cd(x,M_r)^2} \ .
$$

Now $(*_1)$ follows.

The next step is to show that

$$\text{Tr}_s \int_0^1 e^{-(1-s)A(\rho)_t^2} \mathcal{R} e^{-sA(\rho)_t^2} ds = \text{Tr}_s \mathcal{R} e^{-A(\rho)_t^2}$$

or equivalently

$$\int_M \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \text{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \; dy ds dx$$

$$= \int_0^1 (1-s)^{m/2} s^{n/2} \int_M \int_M \text{tr}_s p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n \; dx dy ds \; .$$

We can interchange the integration over s and x by the estimate $(*_2)$.

Fix s and t. Consider once more the decomposition of

$$p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) p(\rho)_{st}(y,x)^n$$

from above. From Cor. 4.3.2 it follows that for $\varepsilon > 0$ there are $r, c, C > 0$ such that for all $x, y \in M$

$$|p(\rho)_{(1-s)t}(x,y)^m \mathcal{R}(y) \big(p(\rho)_{st}(y,x)^n - w(\rho)_{st}(y,x)^n \big)|$$

$$\leq \; C \big(e^{-cd(x,y)^2} + e^{-cd(x,M_r)^2} \big) e^{-cd(y,M_r)^2} \; .$$

The second term can be estimated in an analogous manner and the supertrace of the third term vanishes as we saw.

Hence we can interchange dx and dy.

This shows (1) by

$$\text{Tr}_s \frac{d}{dt} \; e^{-A(\rho)_t^2} \;\; = \;\; \text{Tr}_s (-t^{-1} A(\rho)_t^2 + \frac{1}{2} t^{-1/2} \mathcal{R}) e^{-A(\rho)_t^2}$$

$$= \;\; -\text{Tr}_s \frac{dA(\rho)_t^2}{dt} \; e^{-A(\rho)_t^2} \; .$$

(2) As above, since $A(\rho)_t^2 = t(D^2 + \rho[D,K]_s + \rho^2 K^2) + \sqrt{t}\mathcal{R}$, Duhamel's formula (Prop. 5.4.4) and the chain rule imply that

$$\frac{d}{d\rho} \; e^{-A(\rho)_t^2} = -\int_0^1 e^{-(1-s)A(\rho)_t^2} \frac{dA(\rho)_t^2}{d\rho} \; e^{-sA(\rho)_t^2} \; ds \; .$$

Note that $\frac{dA(\rho)_t^2}{d\rho}$ is a trace class operator, hence we conclude immediately that

$$\frac{d}{d\rho} \; \text{Tr}_s e^{-A(\rho)_t^2} = \text{Tr}_s \frac{d}{d\rho} \; e^{-A(\rho)_t^2} = -\text{Tr}_s \frac{dA(\rho)_t^2}{d\rho} \; e^{-A(\rho)_t^2} \; .$$

Since $A(\rho)_t$ is a bounded perturbation of $\sqrt{t} W D_s W^*$, we have, by Cor. 3.5.8 and Prop. 5.4.10, that $A(\rho)_t$ and $e^{-A(\rho)_t^2}$ commute.

It follows that

$$\frac{dA(\rho)_t^2}{d\rho} e^{-A(\rho)_t^2} \;\; = \;\; [A(\rho)_t, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s$$

$$= \;\; [W^* \, \text{d} \, W, \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s + \sqrt{t} [D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s \; .$$

The first term of the last line is an integral operator with integral kernel $W^* d(k) W$ where k is the integral kernel of $W \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} W^*$. Hence the supertrace of the

first term equals

$$\mathrm{d}\,\mathrm{Tr}_s \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2} \ .$$

Now consider the second term. Let P be the orthogonal projection onto the range of K. It is an even Hilbert-Schmidt operator with smooth complex integral kernel. Since $\frac{dA(\rho)_t}{d\rho} = \sqrt{t}K$,

$$\mathrm{Tr}_s[D(\rho), \frac{dA(\rho)_t}{d\rho} e^{-A(\rho)_t^2}]_s = \mathrm{Tr}_s \sqrt{t}[D(\rho)P, Ke^{-A(\rho)_t^2}]_s \ .$$

Since $D(\rho)P$ and $Ke^{-A(\rho)_t^2}$ are Hilbert-Schmidt operators, the supertrace vanishes.
$\square$

Let D_{I_k} be the operator D_I from §3.3.1 with boundary conditions given by the pair $(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3})$.

In §4.3.2 we defined $A_t^{Z_k} = \tilde{W}_k^* \, \mathrm{d}\, \tilde{W}_k + \sqrt{t} D_{Z_k}$ such that $A_t^{Z_k}$ coincides with $A(\rho)_t$ on $\mathcal{U}_k$. There is $U_k \in C^\infty([0,1], M_{2d}(\mathcal{A}_\infty))$ such that $\tilde{W}_k(x_1, x_2) = U_k(x_2) \oplus U_k(x_2)$. Let $A_t^{I_k} = U_k^* \, \mathrm{d}\, U_k + \sqrt{t} \sigma D_{I_k}$.

LEMMA 4.4.10.

$$\frac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2} = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}/6} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} - \frac{1}{2\sqrt{t}} \mathrm{d}\,\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \ .$$

PROOF. Since $\frac{dA(\rho)_t^2}{dt} = [\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s$, it follows from the previous lemma that

$$\frac{d}{dt} \mathrm{Tr}_s e^{-A(\rho)_t^2} = -\mathrm{Tr}_s[\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2} \ .$$

Furthermore

$$-[\frac{dA(\rho)_t}{dt}, A(\rho)_t]_s e^{-A(\rho)_t^2}$$
$$= -[A(\rho)_t, \frac{dA(\rho)_t}{dt} e^{-A(\rho)_t^2}]_s$$
$$= -\frac{1}{2\sqrt{t}}[W^* \, \mathrm{d}\, W, D(\rho) e^{-A(\rho)_t^2}]_s - \frac{1}{2}[D(\rho), D(\rho) e^{-A(\rho)_t^2}]_s \ .$$

The supertrace of the first supercommutator in the last line equals

$$-\frac{1}{2\sqrt{t}} \mathrm{d}\,\mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \ .$$

Now consider the second term.

We have that

$$-\frac{1}{2} \mathrm{Tr}_s[D(\rho), D(\rho) e^{-A(\rho)_t^2}]_s$$
$$= -\frac{1}{2} \lim_{r \to \infty} \left(\mathrm{Tr}_s \phi_r D(\rho)^2 e^{-A(\rho)_t^2} \phi_r + \mathrm{Tr}_s \phi_r D(\rho) e^{-A(\rho)_t^2} D(\rho) \phi_r \right) \ .$$

Since for $\nu \in \mathbb{N}_0$ the operators $\phi_r D(\rho)^\nu e^{-A(\rho)_t^2/2}$ and $e^{-A(\rho)_t^2/2}D(\rho)^\nu \phi_r$ are Hilbert-Schmidt operators, it follows that

$$
-\frac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s
$$
$$
= -\frac{1}{2}\lim_{r\to\infty}(\mathrm{Tr}_s e^{-A(\rho)_t^2/2}\phi_r^2 D(\rho)^2 e^{-A(\rho)_t^2/2} - \mathrm{Tr}_s e^{-A(\rho)_t^2/2}D(\rho)\phi_r^2 D(\rho)e^{-A(\rho)_t^2/2})
$$
$$
= \frac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s e^{-A(\rho)_t^2/2}c(d\phi_r^2)D(\rho)e^{-A(\rho)_t^2/2}
$$
$$
= \frac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r^2)D(\rho)e^{-A(\rho)_t^2} .
$$

For $r > 0$ we define the function $\chi_r : Z \to \mathbb{R}$, $\chi_r(x) := \chi^2(x_1 - r)$ where χ is the function from the beginning of §4.4.1. Cor. 4.3.3 implies that

$$
\frac{1}{2}\lim_{r\to\infty}\mathrm{Tr}_s c(d\phi_r^2)D(\rho)e^{-A(\rho)_t^2} = \frac{1}{2}\lim_{r\to\infty}\sum_{k\in\mathbb{Z}/6}\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} .
$$

Recall from §4.2 that the integral kernel of $e^{-(A_t^{Z_k})^2}$ is $\sum_{n=0}^{\infty}(-1)^n t^{\frac{n}{2}}p_t^{Z_k}(x,y)^n$ with

$$
p_t^{Z_k}(x,y)^n = c(dx_1)^n \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_1-y_1)^2}{4t}}\left(p_t^{I_k}(x_2,y_2)^n \oplus (-1)^n p_t^{I_k}(x_2,y_2)^n\right) .
$$

The integral kernel of $c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2}$ is

$$
\sum_{n=0}^{\infty}(-1)^{n+1}t^{n/2}\chi'(x_1 - r)(\partial_{x_1} + I\partial_{x_2})p_t^{Z_k}(x,y)^n .
$$

It follows that

$$
\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2}
$$
$$
= \frac{1}{\sqrt{4\pi t}}\sum_{n=0}^{\infty}(-1)^n t^{n/2}\int_0^1 \mathrm{tr}_s\, c(dx_1)^n\left((D_{I_k}p_t^{I_k})(x_2,x_2)^n \oplus\right.
$$
$$
\left. \oplus(-1)^{n+1}(D_{I_k}p_t^{I_k})(x_2,x_2)^n\right)\,dx_2 .
$$

Comparison with Def. 4.1.6 and the subsequent remark yields that

$$
\mathrm{tr}_s c(dx_1)^n\left((D_{I_k}p_t^{I_k})(x_2,x_2)^n \oplus (-1)^{n+1}(D_{I_k}p_t^{I_k})(x_2,x_2)^n\right)
$$
$$
= 2i^n \mathrm{tr}_\sigma \sigma^n (D_{I_k}p_t^{I_k})(x_2,x_2)^n .
$$

It follows that

$$
\mathrm{Tr}_s c(d\chi_r)D_{Z_k}e^{-(A_t^{Z_k})^2} = \frac{2}{\sqrt{4\pi t}}\sum_{n=0}^{\infty}(-1)^n t^n \int_0^1 \mathrm{tr}\,(D_{I_k}p_t^{I_k})(x_2,x_2)^{2n}dx_2
$$
$$
= \frac{1}{\sqrt{\pi t}}\mathrm{Tr}_\sigma D_{I_k}e^{-(A_t^{I_k})^2} ,
$$

hence

$$
-\frac{1}{2}\mathrm{Tr}_s[D(\rho), D(\rho)e^{-A(\rho)_t^2}]_s = \sum_{k\in\mathbb{Z}/6}\frac{1}{\sqrt{4\pi t}}\mathrm{Tr}_\sigma D_{I_k}e^{-(A_t^{I_k})^2} .
$$

$\square$

4.4.5. The index theorem. In this section the notation is as in Prop. 2.5.1. Let A_{I_k} be as in the previous section.

THEOREM 4.4.11.

$$\text{ch}(\text{ind}\, D^+) = \frac{1}{2}[\sum_{k\in\mathbb{Z}/6} \eta(A_{I_k})] \in H_{ev}^{dR}(\mathcal{A}_\infty) \ .$$

Here we understand $\text{ind}\, D^+$ as a class in $K_0(\mathcal{A}_\infty)$ via the isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty)$ induced by the injection $\mathcal{A}_\infty \hookrightarrow \mathcal{A}$.

PROOF. Let $\rho \neq 0$.
By Prop. 2.5.1

$$\text{ind}\, D^+ = [\text{Ker}\, D(\rho)^2] - [\mathcal{A}^{\mathcal{N}}]$$

in $K_0(\mathcal{A})$. Let P_0 be the projection onto the kernel of $D(\rho)^2$.
From Prop. 3.5.5 and Prop. 5.3.6 it follows that

$$\text{ind}\, D^+ = [\text{Ran}_\infty\, P_0] - [\mathcal{A}_\infty^{\mathcal{N}}] = [\text{Ran}_\infty\, WP_0W^*] - [\mathcal{A}_\infty^{\mathcal{N}}]$$

in $K_0(\mathcal{A}_\infty)$, hence in $H_*^{dR}(\mathcal{A}_\infty)$

$$\text{ch}(\text{ind}\, D^+) = \text{ch}[\text{Ran}_\infty\, WP_0W^*] - \mathcal{N}.$$

By Prop. 5.3.6 in $H_*^{dR}(\mathcal{A}_\infty)$

$$
\begin{aligned}
\text{ch}[\text{Ran}_\infty\, WP_0W^*] &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\text{Tr}_s(WP_0W^*\,\text{d}\,WP_0W^*)^{2n} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\text{Tr}_s(P_0W^*\,\text{d}\,WP_0)^{2n} \ .
\end{aligned}
$$

Let $T > 0$. By Th. 4.4.3 and Th. 4.4.8 in $\hat{\Omega}_*\mathcal{A}_\infty/\overline{[\hat{\Omega}_*\mathcal{A}_\infty,\hat{\Omega}_*\mathcal{A}_\infty]_s}$

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\text{Tr}_s(P_0W^*\,\text{d}\,WP_0)^{2n} - \mathcal{N} &= \lim_{t\to\infty}\text{Tr}_s e^{-A(\rho)_t^2} - \lim_{t\to 0}\text{Tr}_s e^{-A_t^2} \\
&= \int_T^{\infty}\frac{d}{dt}\text{Tr}_s e^{-A(\rho)_t^2}\,dt \\
&\quad + \int_0^{\rho}\frac{d}{d\rho'}\text{Tr}_s e^{-A(\rho')_T^2}\,d\rho' \\
&\quad + \int_0^T\frac{d}{dt}\text{Tr}_s e^{-A_t^2}\,dt \ .
\end{aligned}
$$

By the results of §4.4.4, the estimate of Th. 4.4.3 and Th. 4.4.8 the integrals converge and we have:

$$
\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}_s (P_0 W^* \, \mathrm{d}\, W P_0)^{2n} - \mathcal{N} \;=\;& \sum_{k \in \mathbb{Z}/6} \int_0^{\infty} \frac{1}{\sqrt{4\pi t}} \mathrm{Tr}_\sigma D_{I_k} e^{-(A_t^{I_k})^2} \, dt \\
& - \mathrm{d} \int_T^{\infty} \frac{1}{2\sqrt{t}} \mathrm{Tr}_s D(\rho) e^{-A(\rho)_t^2} \, dt \\
& - \mathrm{d} \int_0^{\rho} \mathrm{Tr}_s \frac{dA(\rho')_T}{d\rho'} e^{-A(\rho')_T^2} \, d\rho' \\
& - \mathrm{d} \int_0^T \frac{1}{2\sqrt{t}} \mathrm{Tr}_s D(0) e^{-A_t^2} \, dt \; .
\end{aligned}
$$

The assertion follows.   □

COROLLARY 4.4.12. *In* $H_*^{dR}(\mathcal{A}_\infty)$

$$
\mathrm{ch}\, \tau(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = [\eta(\mathcal{P}_0, \mathcal{P}_1) + \eta(\mathcal{P}_1, \mathcal{P}_2) + \eta(\mathcal{P}_2, \mathcal{P}_0)] \; .
$$

PROOF. For $\rho > 0$ we define a family $A(\rho, b)$ of superconnections associated to $D(\rho)$. For $0 < b < b_0$ let $W(b) \in C^\infty(M, \mathrm{End}^+ E \otimes \mathcal{A}_\infty)$ be as in §4.3.1 such that $W(b)$ is parallel on $\{x \in M \mid d(x, \partial M) \geq b\}$. As in §4.3.1 we set

$$
A(\rho, b) := W(b)^* \, \mathrm{d}\, W(b) + D(\rho) \; .
$$

For every $k \in \mathbb{Z}/6$ this induces a family of superconnections $A_{I_k}(b)$. By Prop. 4.1.7 and the subsequent definition

$$
\lim_{b \to 0} \eta(A_{I_k}(b)) = \eta(\mathcal{P}_{k \bmod 3}, \mathcal{P}_{k+1 \bmod 3}) \; .
$$

The assertion follows now by the previous theorem and Prop. 2.4.4.   □

4.5. A gluing formula for η-forms on S^1

In the following we sketch a reinterpretation of Cor. 4.4.12 as a gluing formula for η-forms on S^1 (which we identify with $\mathbb{R}/\mathbb{Z}$ as a Riemannian manifold).

Given $u \in U(\mathcal{A}_\infty^n)$ we define a projective system of vector bundles on S^1:

$$
\mathcal{L}_i(u) = ([0, 1] \times \mathcal{A}_i^n)/\sim
$$

with $(0, v) \sim (1, uv)$. The standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^n$ induces a hermitian $\mathcal{A}$-valued metric on $\mathcal{L}(u) := \mathcal{L}_0(u)$. We identify a smooth section of $\mathcal{L}_i(u)$ with a smooth function $f : \mathbb{R} \to \mathcal{A}_i^n$ satisfying $f(x + 1) = uf(x)$. The trivial connection $f \mapsto f' dx$ on $\mathbb{R} \times \mathcal{A}^n$ induces a connection on $\mathcal{L}(u)$ preserving the metric. The associated Dirac operator is denoted by $\eth_{\mathcal{L}(u)}$ (as are its closures on the spaces $L^2(S^1, \mathcal{L}_i(u))$ in the following).

Assume now $1 \notin \sigma(u)$. Then there is a path $w : [0, 1] \to U(\mathcal{A}_\infty^n)$ with $w(0) = 1$, $w(1) = u$, and the map $f \mapsto w^* f$ defines a trivialization of $\mathcal{L}_i(u)$. Hence $\eth_{\mathcal{L}(u)}$ can be identified with the operator $i\partial + w^* w'$ on the trivial bundle. One can easily deduce that $-\eth_{\mathcal{L}(u)}^2$ generates a holomorphic semigroup on $L^2(S^1, \mathcal{L}_i(u))$ with integral kernel in $C^\infty(S^1 \times S^1, M_n(\mathcal{A}_\infty))$. Since $\eth_{\mathcal{L}(u)}^2$ is invertible on $L^2(S^1, \mathcal{L}_i(u))$, the integral kernel vanishes exponentially for $t \to \infty$. Locally there is a trivialization of $\mathcal{L}(u)$ with respect to which $\eth_{\mathcal{L}(u)}$ equals $i\partial$. By Duhamel's principle it follows that $\mathrm{Tr}\, \eth_{\mathcal{L}(u)} e^{-t(\eth_{\mathcal{L}(u)})^2}$ converges to zero in $\mathcal{A}_\infty/\overline{[\mathcal{A}_\infty, \mathcal{A}_\infty]}$ for $t \to 0$.

We define a superconnection $A := w \, \mathrm{d} \, w^* + \slashed{\partial}_{\mathcal{L}(u)}$ and the associated rescaled superconnection $A_t := w \, \mathrm{d} \, w^* + \sqrt{t} \slashed{\partial}_{\mathcal{L}(u)}$. Then $e^{-A_t^2}$ is a well-defined integral operator with smooth integral kernel. Furthermore $\mathrm{Tr} \, \slashed{\partial}_{\mathcal{L}(u)} e^{-A_t^2}$ has a limit for $t \to 0$, and the η-form

$$\eta(A_w) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \, \mathrm{Tr} \, \slashed{\partial}_{\mathcal{L}(u)} e^{-A_t^2} \, dt \in \hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s}$$

is well-defined. As in Def. 4.1.8 we can eliminate the dependence of the trivialization w by defining

$$\eta(\slashed{\partial}_{\mathcal{L}(u)}) = \lim_{\varepsilon \to 0} \eta(A_{w_\varepsilon})$$

where w_ε is a family of trivializations with $w'_\varepsilon \subset [0, \varepsilon] \cup [1 - \varepsilon, 1]$. The existence of the limit follows from the proof of the next proposition.

In the following let

$$P(u) := \frac{1}{2} \begin{pmatrix} 1 & u^* \\ u & 1 \end{pmatrix} .$$

PROPOSITION 4.5.1. *Let u_0, u_1 be unitaries such that $u_0 - u_1$ is invertible. Then up to exact forms*

$$\eta(\slashed{\partial}_{\mathcal{L}(u_0^* u_1)}) = \eta(P(u_0), P(u_1)) .$$

PROOF. For a unitary $U \in M_n(\mathcal{A}_\infty)$

$$\eta(U^* P(u_0) U, U^* P(u_1) U) = \eta(P(u_0), P(u_1)) .$$

Since for $U := \begin{pmatrix} 1 & 0 \\ 0 & u_0 \end{pmatrix}$ we have that $U^* P(u_0) U = P(1)$ and $U^* P(u_1) U = P(u_0^* u_1)$, we may assume $u_0 = 1$. Let $u := u_1$.

Let $V : \mathbb{R} \to U(\mathcal{A}_\infty^n)$ be a smooth path with $V(x) = u$ for $x \le 0$ and $V(x) = -1$ for $x \ge 1$ and with $1 \notin \sigma(V)$; we define the bundle

$$L_i(V) := (\mathbb{R} \times [0, 1]) \times \mathcal{A}_i^n / \sim$$

where $(x_1, 0, v) \sim (x_1, 1, V(x_1) v)$. The smooth sections of $L_i(V)$ can be identified with smooth functions $f : \mathbb{R} \times \mathbb{R} \to \mathcal{A}^n$ fulfilling $f(x_1, x_2 + 1) = V(x_1) f(x_1, x_2)$. Let $W : \mathbb{R} \times [0, 1] \to U(\mathcal{A}_\infty^n)$ be smooth with $W(x_1, 0) = 1$, $W(x_1, 1) = V(x_1)$ and such that $W(x_1, x_2)$ is independent of x_2 in a small neighborhood of 0 resp. 1 and independent of x_1 for $x_1 < 0$ resp. $x_1 > 1$. Then $f \mapsto W^* f$ is a bundle isomorphism between $L_i(V)$ and the trivial bundle $(\mathbb{R} \times S^1) \times \mathcal{A}_i^n$. Since $dx_1 \partial_1 + W^* dx_2 \partial_2 W = dx_1 \partial_1 + dx_2 \partial_2 + dx_2 W^* (\partial_2 W)$ is a connection preserving the metric on the trivial bundle, the operator $W dx_1 \partial_1 W^* + dx_2 \partial_2$ is a connection preserving the metric on $L_i(V)$. It agrees with the trivial connection for $|x_1|$ large.

Let $\slashed{\partial}_{L_i(V)}$ be the Dirac operator associated to the connection $W dx_1 \partial_1 W^* + dx_2 \partial_2$ on the bundle $L_i(V)$. The index of $\slashed{\partial}_{L_i(V)}^+$ vanishes, which can be seen as follows. For $t \in [0, 1]$ let $W_t : \mathbb{R} \times [0, 1] \to U(\mathcal{A}_\infty^n)$ be a homotopy with $W_0 = 1$ and $W_1 = W$ and independent of x_1 on the complement of a compact set. Then $dx_1 \partial_1 + W_t^* dx_2 \partial_2 W_t$ is a homotopy of connections on the trivial bundle interpolating between the trivial connection and the connection $dx_1 \partial_1 + W^* dx_2 \partial_2 W$. The index of the Dirac operator associated to the trivial connection vanishes, hence the index of the Dirac operator associated to $dx_1 \partial_1 + W^* dx_2 \partial_2 W$ vanishes as well.

The local term in the index theorem is determined by the superconnection

$$B_1 := \mathrm{d} + W \slashed{\partial}_{L_i(V)} W^* = \mathrm{d} + c(dx_1) \partial_1 + c(dx_2) \partial_2 + c(dx_2) W (\partial_2 W^*)$$

on the trivial bundle $S \otimes \mathcal{A}^n$ on $\mathbb{R} \times S^1$. The contribution from the cylindric ends is given by $\frac{1}{2}(\eta(\partial\!\!\!/_{\mathcal{L}(-1)}) - \eta(A))$.

Compare this with the index theorem for the Dirac operator $\partial\!\!\!/_Z$ on the sections of $(\mathbb{R} \times [0,1]) \times ((\mathcal{A}^+)^{2n} \oplus (\mathcal{A}^-)^{2n})$ with boundary conditions defined by $P(1) \oplus P(1)$ and $P(V(x_1)) \oplus P(V(x_1))$. Let

$$\tilde{W} := \mathrm{diag}(1, W, 1, W) \in M_{4n}(\mathcal{A}_\infty) \ .$$

Then

$$\tilde{W}^* \partial\!\!\!/_Z \tilde{W} = \partial\!\!\!/_Z + \tilde{W}^* c(dx_1)(\partial_1 \tilde{W}) + \tilde{W}^* c(dx_2)(\partial_2 \tilde{W}) \ ,$$

and the boundary conditions transform to $P(1) \oplus P(1)$ and $P(-1) \oplus P(-1)$.

An argument analogous to the one above shows that the index of $\tilde{W}^* \partial\!\!\!/_Z^\pm \tilde{W}$ vanishes. Since $\tilde{W}^* c(dx_1)(\partial_1 \tilde{W})$ is compactly supported, the index of $\tilde{W}^* \partial\!\!\!/_Z \tilde{W}$ equals the index of $\partial\!\!\!/_Z + \tilde{W}^* c(dx_2)(\partial_2 \tilde{W})$, which is the Dirac operator associated to the connection $dx_1 \partial_1 + dx_2 \partial_2 + dx_2 \tilde{W}^* (\partial_2 \tilde{W})$. Hence we can use the superconnection $B_2 := \mathrm{d} + c(dx_1)\partial_1 + c(dx_2)\partial_2 + c(dx_2)\tilde{W}^*(\partial_2 \tilde{W})$ for the index theorem, and it follows that the local term here equals the local term in the index theorem for $\partial\!\!\!/_{L(V)}$. Since the proposition is true for $u = -1$ by [**CLM**], Prop. 6.3, the assertion follows. $\qquad \qquad \square$

COROLLARY 4.5.2. *If $u_0, u_1, u_2 \in M_n(\mathcal{A}_\infty)$ are unitaries such that $u_i - u_j$ is invertible for $i \neq j$, then*

$$\mathrm{ch}\,\tau(P(u_0), P(u_1), P(u_2)) = [\eta(\partial\!\!\!/_{\mathcal{L}(u_0^* u_1)}) + \eta(\partial\!\!\!/_{\mathcal{L}(u_1^* u_2)}) + \eta(\partial\!\!\!/_{\mathcal{L}(u_2^* u_0)})] \in H_*^{dR}(\mathcal{A}_\infty) \ .$$

Assume that $P_i - P_j$ and $P_i - (1 - P_j)$ are invertible for $i \neq j$ and define

$$\begin{aligned} \tau_I(P_0, P_1, P_2) &:= \tau(P_0, P_1, P_2) + \tau(P_0, 1 - P_1, P_1) \\ &\quad + \tau(P_1, 1 - P_2, P_2) + \tau(P_2, 1 - P_0, P_0) \in K_0(\mathcal{A}) \ . \end{aligned}$$

Then from the corollary, the previous formula and the fact that $1 - P(u) = P(-u)$ it follows after some straightforward calculations:

COROLLARY 4.5.3. *In $H_*^{dR}(\mathcal{A}_\infty)$*

$$\mathrm{ch}\,\tau_I(P(u_0), P(u_1), P(u_2)) = [\eta(\partial\!\!\!/_{\mathcal{L}(-u_0^* u_1)}) + \eta(\partial\!\!\!/_{\mathcal{L}(-u_1^* u_2)}) + \eta(\partial\!\!\!/_{\mathcal{L}(-u_2^* u_0)})] \ .$$

In the following we explain how this formula is related to a gluing formula for η-invariants ([**Bu**], §1.7 and Cor. 1.20).

Consider the Clifford bundle $\mathcal{A}^n \times [0, \pi]$ on the manifold with boundary $[0, \pi]$ with Clifford multiplication i and Dirac operator $i\partial$. Its restriction to the boundary is $\mathcal{A}^n \times (\{0\} \cup \{\pi\})$ and the Clifford multiplication by the inward pointing normal vector is given by the bundle morphism $I_0 = i \cup (-i)$. Via the standard $\mathcal{A}$-valued scalar product it induces a skew-hermitian form on the bundle $\mathcal{A}^n \times (\{0\} \cup \{\pi\})$. We identify the sections of this bundle with $\mathcal{A}^{2n}$. The image of the kernel of the Dirac operator with respect to restriction to the boundary is the Lagrangian submodule $L = \{(x, x) \mid x \in \mathcal{A}^n\}$.

Now let

$$\phi_j := (1 \cup (u_j)) : \mathcal{A}^n \times (\{0\} \cup \{\pi\}) \to \mathcal{A}^n \times (* \cup *) \ .$$

Then $\phi_j(L)$ is the range of the projection $P(u_j)$.

The bundle $\mathcal{L}(-u_j^* u_{j+1})$ on S^1 can be obtained by gluing

$$(\mathcal{A}^n \times [0, \pi]) \cup_{\phi_j^{-1} \circ I_0 \circ \phi_{j+1}} (\mathcal{A}^n \times [0, \pi]) \ ;$$

and by gluing the operator $i\partial$ on one copy of $[0, \pi]$ with $-i\partial$ on the second copy one obtains $\bar{\partial}_{\mathcal{L}(-u_0^* u_1)}$.

From the proposition it follows that up to exact forms

$$\eta(\bar{\partial}_{\mathcal{L}(-u_j^* u_{j+1})}) = \eta(P(u_j), 1 - P(u_{j+1})) \ .$$

Since the projection $\phi_i(L)$ in $\mathcal{A}^{2n}$ is given by the projection $P(u_j)$, we have in $H_*^{dR}(\mathcal{A}_\infty)$:

$$\operatorname{ch} \tau_I(\phi_0(L), \phi_1(L), \phi_2(L)) = [\eta(\bar{\partial}_{\mathcal{L}(-u_0^* u_1)}) + \eta(\bar{\partial}_{\mathcal{L}(-u_1^* u_2)}) + \eta(\bar{\partial}_{\mathcal{L}(-u_2^* u_0)})] \ .$$

For $\mathcal{A} = \mathbb{C}$ this is a particular case of the well-known general gluing formula.

CHAPTER 5

Definitions and Techniques

5.1. Hilbert C^*-modules

5.1.1. Bounded operators. Let $\mathcal{A}$ be a unital C^*-algebra with norm $|\cdot|$.

In order to fix notation we recall some facts about Hilbert $\mathcal{A}$-modules. References are [**La**] and [**WO**].

DEFINITION 5.1.1. *A* PRE-HILBERT $\mathcal{A}$-MODULE *is a right $\mathcal{A}$-module H with an $\mathcal{A}$-valued scalar product $\langle\ ,\ \rangle : H \times H \to \mathcal{A}$; i.e.*

(1) $\langle\ ,\ \rangle$ *is $\mathcal{A}$-linear in the second variable,*
(2) $\langle x, y \rangle = \langle y, x \rangle^*$ *for all $x, y \in H$,*
(3) $\langle x, x \rangle \geq 0$ *for all $x \in H$,*
(4) *if $\langle x, x \rangle = 0$, then $x = 0$.*

If H is complete with respect to the norm $\|v\| := |\langle v, v \rangle|^{1/2}$, then H is called a HILBERT $\mathcal{A}$-MODULE.

The completion of a pre-Hilbert $\mathcal{A}$-module is a Hilbert $\mathcal{A}$-module.

The right $\mathcal{A}$-module $\mathcal{A}^n$ is a Hilbert $\mathcal{A}$-module when endowed with the standard $\mathcal{A}$-valued scalar product

$$\langle a, b \rangle := \sum_{i=1}^{n} a_i^* b_i \ .$$

The right $\mathcal{A}$-module $\{(a_n)_{n\in\mathbb{N}} \subset \mathcal{A} \mid \sum_{n=1}^{\infty} a_n^* a_n \text{ converges}\}$ endowed with the $\mathcal{A}$-valued scalar product

$$\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \rangle := \sum_{n=1}^{\infty} a_n^* b_n$$

is a Hilbert $\mathcal{A}$-module and is denoted by $l^2(\mathcal{A})$. Sometimes we use $\mathbb{Z}$ as index set.

Let M be a measure space and let $\langle\ ,\ \rangle$ be the standard $\mathcal{A}$-valued scalar product on $\mathcal{A}^n$. Then the Hilbert $\mathcal{A}$-module $L^2(M, \mathcal{A}^n)$ is defined in the following way: By

$$\langle f, g \rangle_{L^2} = \int_M \langle f(x), g(x) \rangle dx$$

an $\mathcal{A}$-valued scalar product is defined on the quotient of the space of simple functions on M with values in $\mathcal{A}^n$ by the space of simple functions vanishing almost everywhere. Hence the quotient is a pre-Hilbert $\mathcal{A}$-module. Its completion is the Hilbert $\mathcal{A}$-module $L^2(M, \mathcal{A}^n)$.

Let H be a Hilbert $\mathcal{A}$-module.

A submodule $U \subset H$ is called complemented if

$$U^\perp = \{x \in H \mid \langle x, u \rangle = 0\ \forall u \in U\}$$

satisfies $U \oplus U^\perp = H$. Any projective submodule in H is complemented.

Let H_1, H_2 be Hilbert $\mathcal{A}$-modules. The elements of

$$B(H_1, H_2) \quad = \quad \{T : H_1 \to H_2 \mid T \text{ is continuous and } \exists T^* : H_2 \to H_1 \text{ with}$$
$$\langle Tv, w \rangle_{H_2} = \langle v, T^*w \rangle_{H_1} \ \forall v \in H_1, \ w \in H_2\}$$

are called bounded operators from H_1 to H_2. They form a Banach space with respect to the operator norm. With the composition as a product $B(H_1) :=$ $B(H_1, H_1)$ is a C^*-algebra. Note that the existence of an adjoint must be required.

A continuous $\mathcal{A}$-module map $K : H_1 \to H_2$ is called compact if it can be approximated by a linear combination of operators of the form $x \mapsto z\langle y, x\rangle$ with $y \in H_1, z \in H_2$, in the operator norm topology.

Every compact operator is adjointable.

A projection in $B(H)$ is compact if and only if it is a projection onto a projective submodule of H.

If the range of $T \in B(H_1, H_2)$ is complemented, we call its complement the cokernel $\mathrm{Coker}\, T$. Clearly a necessary condition for its existence is that the range of T is closed. The following proposition shows that it is sufficient:

PROPOSITION 5.1.2. *Suppose that $T \in B(H_1, H_2)$ has closed range. Then*
  (1) $\mathrm{Ker}\, T$ *is a complemented submodule of H_1,*
  (2) $\mathrm{Ran}\, T$ *is a complemented submodule of H_2,*
  (3) $T^* : H_2 \to H_1$ *also has closed range.*

PROOF. [**La**], Th. 3.2. $\square$

5.1.2. Fredholm operators. Let H_1, H_2 be Hilbert $\mathcal{A}$-modules isomorphic to $l^2(\mathcal{A})$.

DEFINITION 5.1.3. *An operator $F \in B(H_1, H_2)$ is* FREDHOLM *if there are decompositions $H_1 = M_1 \oplus N_1$ and $H_2 = M_2 \oplus N_2$ with the following properties:*
  (1) N_1, N_2 *are projective $\mathcal{A}$-modules and M_1, M_2 are closed $\mathcal{A}$-modules.*
  (2) *The operator F is diagonal: $F = F_M \oplus F_N$ with $F_M : M_1 \to M_2$ and $F_N : N_1 \to N_2$.*
  (3) *The component $F_M : M_1 \to M_2$ is an isomorphism.*
  (4) *The projection onto N_i along M_i is adjointable for $i = 1, 2$.*
The INDEX OF F *is defined as*

$$\mathrm{ind}\, F := [N_1] - [N_2] \in K_0(\mathcal{A}) .$$

PROPOSITION 5.1.4. *A selfadjoint operator $F \in B(H_1, H_2)$ is Fredholm if and only if there exists an $\mathcal{A}$-linear continuous, not necessarily adjointable, map $G :$ $H_2 \to H_1$ such that $FG - 1$ and $GF - 1$ are compact.*

PROOF. Analogous to [**MF**], Theorem 2.4. $\square$

From the proposition it follows that if F is Fredholm, then for any compact operator K the operator $F + K$ is Fredholm.

PROPOSITION 5.1.5. *If $F : H_1 \to H_2$ is a Fredholm operator and $K : H_1 \to H_2$ is a compact operator, then*

$$\mathrm{ind}\, F = \mathrm{ind}(F + K) .$$

PROOF. [**MF**], Lemma 2.3. $\square$

Another important property of Fredholm operators is the following:

PROPOSITION 5.1.6. *If $F \in B(H_1, H_2)$ is Fredholm and $\operatorname{Ran} F$ is closed, then $\operatorname{Ker} F$ and $\operatorname{Coker} F$ are projective modules.*
Hence

$$\operatorname{ind} F = [\operatorname{Ker} F] - [\operatorname{Coker} F] \ .$$

PROOF. Let $P_{\operatorname{Ker} F} \in B(H_1)$ resp. $P_{\operatorname{Coker} F} \in B(H_2)$ be the orthogonal projection onto the kernel resp. cokernel of F. They exist by Prop. 5.1.2. We have to prove that they are compact.
There is $E \in B(H_2, H_1)$ such that $EF = 1 - P_{\operatorname{Ker} F}$ and $FE = 1 - P_{\operatorname{Coker} F}$.
Let G be a parametrix of F. It follows that

$$P_{\operatorname{Ker} F} = 1 - EF = (1 - GF)(1 - EF)$$

and

$$P_{\operatorname{Coker} F} = 1 - FE = (1 - FE)(1 - FG)$$

are compact operators. $\square$

PROPOSITION 5.1.7. *If $F : [0,1] \to B(H_1, H_2)$ is a continuous path of Fredholm operators, then the map $[0,1] \to K_0(\mathcal{A})$, $t \mapsto \operatorname{ind} F(t)$ is constant.*

PROOF. see [**WO**], Prop. 17.3.4. $\square$

5.1.3. Regular operators. In this section some basic facts about unbounded operators on Hilbert $\mathcal{A}$-modules are collected. Most of them and more can be found in [**La**].
Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $\langle \ , \ \rangle$. Let $D : \operatorname{dom} D \to H$ be a densely defined operator on H.

LEMMA 5.1.8. *If the adjoint D^* of D is densely defined, then D is closable.*

PROOF. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\operatorname{dom} D$ such that (f_n, Df_n) converges to $(0, f)$ in $H \oplus H$ for $n \to \infty$. Then for every $g \in \operatorname{dom} D^*$

$$\langle f, g \rangle = \lim_{n \to \infty} \langle Df_n, g \rangle = \lim_{n \to \infty} \langle f_n, D^* g \rangle = 0 \ .$$

Since $\operatorname{dom} D^*$ is dense in H, it follows that $f = 0$. $\square$

If D is closed, then

$$\langle f, g \rangle_D := \langle f, g \rangle + \langle Df, Dg \rangle$$

is an $\mathcal{A}$-valued scalar product on $\operatorname{dom} D$, with respect to which $\operatorname{dom} D$ is a Hilbert $\mathcal{A}$-module, denoted by $H(D)$.

LEMMA 5.1.9. *Assume that D is closed.*
  (1) *Suppose that D has a densely defined adjoint D^*. Then $\operatorname{Ker} D^* = (\operatorname{Ran} D)^{\perp}$.*
  (2) $\operatorname{Ker} D$ *is complemented in $H(D)$ if and only if $\operatorname{Ker} D$ is complemented in H.*

PROOF. (1) Since for $f \in \operatorname{Ker} D^*$ and $h \in \operatorname{dom} D$

$$\langle f, Dh \rangle = \langle D^* f, h \rangle = 0 \ ,$$

the $\mathcal{A}$-module $\operatorname{Ker} D^*$ is a submodule of $(\operatorname{Ran} D)^{\perp}$.
For $g \in (\operatorname{Ran} D)^{\perp}$ the $\mathcal{A}$-linear functional

$$\operatorname{dom} D \to \mathcal{A}, \ f \mapsto \langle g, Df \rangle$$

vanishes. Thus $g \in \operatorname{dom} D^*$ and $D^* g = 0$.

(2) For $g \in \operatorname{Ker} D$ and $f \in \operatorname{dom} D$ the conditions $\langle f, g \rangle_D = 0$ and $\langle f, g \rangle = 0$ are equivalent. Hence if $\operatorname{Ker} D$ is complemented in $H(D)$, then $\operatorname{Ker} D$ is complemented in H.

Conversely, if $\operatorname{Ker} D$ is complemented in H, we can decompose $g \in H(D)$ in a sum $g = g_1 + g_2$ with $g_1 \in \operatorname{Ker} D$ and $g_2 \in (\operatorname{Ker} D)^\perp$. From $\operatorname{Ker} D \subset H(D)$ it follows that $g_2 = g - g_1 \in H(D)$, hence $g_2 \in (\operatorname{Ker} D)^{\perp H(D)}$. $\qquad \square$

Recall that D is called regular if it is closed with densely defined adjoint D^* and if $1 + D^* D$ has dense range, or equivalently if it is closed with densely defined adjoint and if its graph is complemented in $H \times H$.

If D is regular, then $1 + D^* D$ has a bounded inverse.

In the following the adjoint of an operator $A \in B(H(D), H)$ is denoted by $A^T \in B(H, H(D))$ in order to distinguish it from the adjoint A^* of A as an unbounded operator on H.

LEMMA 5.1.10. *Assume that D is closed.*

(1) *The operator D is regular if and only if the inclusion $\iota : H(D) \to H$ is in $B(H(D), H)$ and $(1 + D^* D)$ is selfadjoint. Then $\iota^T = (1 + D^* D)^{-1} \in B(H, H(D))$ and $(1 + D^* D)^{-\frac{1}{2}} : H \to H(D)$ is an isometry.*
(2) *Assume that D is regular and selfadjoint. Then $D \in B(H(D), H)$ and $D^T = D(1 + D^2)^{-1}$.*

PROOF. If D is regular, then for $v \in H(D)$ and $w \in H$

$$
\begin{aligned}
\langle \iota v, w \rangle &= \langle v, w \rangle \\
&= \langle v, (1 + D^* D)(1 + D^* D)^{-1} w \rangle \\
&= \langle v, (1 + D^* D)^{-1} w \rangle + \langle Dv, D(1 + D^* D)^{-1} w \rangle \\
&= \langle v, (1 + D^* D)^{-1} w \rangle_D .
\end{aligned}
$$

This shows $\iota^T = (1 + D^* D)^{-1}$.

Now the converse direction:

Let $v \in H$. Then for any $w \in \operatorname{dom}(1 + D^* D)$

$$
\langle v, w \rangle = \langle \iota^T v, w \rangle_D = \langle \iota^T v, (1 + D^* D) w \rangle .
$$

Since $(1 + D^* D)$ is selfadjoint, it follows that $\iota^T v \in \operatorname{dom}(1 + D^* D)$ and $(1 + D^* D) \iota^T v = v$.

This shows that $(1 + D^* D)$ is surjective and that ι^T is a right inverse of $(1 + D^* D)$. Since $(1 + D^* D)$ is bounded below, it is injective as well. It follows that $(1 + D^* D)$ is invertible and ι^T is its inverse.

The remaining parts are immediate.

$\qquad \square$

PROPOSITION 5.1.11. *Let D_0 be a regular selfadjoint operator and assume $D = D_0 + V$ with $V \in B(H)$.*

(1) *Then D is closed.*
(2) *The identity map induces a continuous isomorphism from $H(D_0)$ to $H(D)$.*
(3) *$D \in B(H(D_0), H)$.*
(4) *Suppose that V is selfadjoint. Then D is regular.*

PROOF. (1) From $\operatorname{dom} D^{**} = \operatorname{dom} D_0^{**} = \operatorname{dom} D_0 = \operatorname{dom} D$ it follows that $D = D^{**}$. Thus D is closed.

Assertion (2) follows from the fact that there is $C > 0$ such that for all $f \in H(D_0)$

$$\begin{aligned} \|f\|_D^2 &\leq \|f\|^2 + \|D_0 f\|^2 + \|\langle V f, D_0 f\rangle\| + \|\langle D_0 f, V f\rangle\| + \|V f\|^2 \\ &\leq C(\|f\|^2 + \|D_0 f\|^2) + 2\|V f\|\|D_0 f\| \\ &\leq C\|f\|_{D_0}^2 \ . \end{aligned}$$

We applied Cauchy-Schwarz inequality.

(3) By (2) the operator $D : H(D_0) \to H$ is continuous. By the previous lemma the adjoint of $D = D_0 + V\iota : H(D_0) \to H$ is

$$D^T = D_0(1 + D_0^2)^{-1} + (1 + D_0^2)^{-1} V^* : H \to H(D_0) \ .$$

(4) By (3) the operator $D + i : H(D_0) \to H$ is an adjointable bounded operator. By [**La**], Lemma 9.7, the range of $D + i$ is closed. Thus it is complemented by Prop. 5.1.2. From Lemma 5.1.9 it follows that the cokernel of $D + i$ agrees with the kernel of $D - i$. By [**La**], Lemma 9.7, the operator $D - i$ is injective. It follows that $\operatorname{Coker}(D + i) = \{0\}$. By [**La**], Lemma 9.8, this shows that D is regular. □

It can be shown that the condition of selfadjointness for D_0 and V in statement (4) of the previous proposition can be dropped.

PROPOSITION 5.1.12. *Let D be a regular and selfadjoint operator on H with closed range.*

(1) *The cokernel of D exists and $\operatorname{Ker} D = \operatorname{Coker} D$. In particular $\operatorname{Ker} D$ is complemented.*
(2) *The $\mathcal{A}$-module $\operatorname{dom} D \cap \operatorname{Ran} D$ is dense in $\operatorname{Ran} D$ and $\operatorname{dom} D = \operatorname{Ker} D \oplus (\operatorname{dom} D \cap \operatorname{Ran} D)$, thus $D = 0 \oplus D|_{\operatorname{Ran} D}$ and $D|_{\operatorname{Ran} D}$ is invertible.*

PROOF. (1) Since, by Lemma 5.1.10, $D \in B(H(D), H)$ and since the range of D is closed, Prop. 5.1.2 implies that the range is complemented. Its complement is $\operatorname{Coker} D$. Since D is selfadjoint, $\operatorname{Ker} D = \operatorname{Coker} D$ by Lemma 5.1.9.

(2) Let $P : H \to \operatorname{Ran} D$ be the orthogonal projection. From $(1 - P)(\operatorname{dom} D) \subset \operatorname{Ker} D \subset \operatorname{dom} D$ we conclude that $P(\operatorname{dom} D) \subset \operatorname{dom} D$. The assertion follows because $P(\operatorname{dom} D)$ is dense in $P(H) = \operatorname{Ran} D$. □

We will need the following $\mathbb{Z}/2$-version of the previous proposition:

If $H = H^+ \oplus H^-$ is $\mathbb{Z}/2$-graded, then we call a closed operator D on H even resp. odd if $\operatorname{dom} D$ decomposes in $(\operatorname{dom} D)^+ \oplus (\operatorname{dom} D)^-$ and if the action of D is even resp. odd.

PROPOSITION 5.1.13. *Let H be a $\mathbb{Z}/2$-graded Hilbert $\mathcal{A}$-module and let D be an odd regular selfadjoint operator on H.*

Suppose that $D^+ : (\operatorname{dom} D)^+ \to H^-$ is surjective.

Then the range of D is closed and complemented. Furthermore $\operatorname{Ker} D^+ = \operatorname{Ker} D = \operatorname{Coker} D = \operatorname{Coker} D^-$ and this module is complemented.

PROOF. Since D^+ is surjective, D^- is injective and so $\operatorname{Ker} D^+ = \operatorname{Ker} D$.

Let P_+ be the orthogonal projection onto H^+. Since D is odd, $DP_+ = P_+ D$.

By Lemma 5.1.10 the operator $DP_+ : H(D) \to H$ is adjointable with adjoint $P_+ D(D^2 + 1)^{-1}$. From $P^+ D|_{H(D)^\pm} = D^\pm$ it follows that $D^-(D^2 + 1)^{-1} : H^- \to H(D)^+$ is the adjoint of $D^+ : H(D)^+ \to H^-$.

Since D^+ is surjective, $\operatorname{Ker} D^+$ is complemented in $H(D)^+$ and the range of the adjoint $D^-(D^2+1)^{-1} : H^- \to H(D)^+$ is closed. Furthermore

$$D^-(D^2+1)^{-1} = (D^2+1)^{-\frac{1}{2}} D^- (D^2+1)^{-\frac{1}{2}} \ ,$$

and $(D^2+1)^{-1/2} : H^\pm \to H(D)^\pm$ is an isomorphism by Lemma 5.1.10, hence $\operatorname{Ran} D^-$ is closed, too. $\square$

PROPOSITION 5.1.14. *Let D be a regular selfadjoint operator on H.*

(1) *For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the operator $D - \lambda$ is invertible.*
(2) *Assume that the range of D is closed and let P be the projection onto the kernel of D. Then there is $c > 0$ such that the spectrum of $(D + P)$ is contained in $\mathbb{R}\setminus]-c, c[$ and the spectrum of D is contained in $(\mathbb{R}\setminus]-c, c[) \cup \{0\}$.*

PROOF. This follows from the functional calculus for regular operators ([**La**], Th. 10.9) and from the decomposition in Prop. 5.1.12. $\square$

The following criteria for selfadjointness are useful:

LEMMA 5.1.15. *Let D be a symmetric regular operator such that the range of $D + i$ and of $D - i$ is dense in H. Then D is selfadjoint.*

PROOF. The operators $D + i$ and of $D - i$ have closed range (see [**La**], Lemma 9.7). It follows that they have a bounded inverse on H. Then they are adjoint to each other, thus D is selfadjoint. $\square$

LEMMA 5.1.16. *Assume that D is symmetric and has an inverse $D^{-1} \in B(H)$. Then D is regular.*

PROOF. Since D is symmetric, the adjoint is densely defined. From $D^{-1} \in B(H)$ it follows that the graph of D^{-1} is complemented, hence the graph of D is complemented as well. Hence D is regular. $\square$

5.1.4. Decompositions of Hilbert C^*-modules. Let H be a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued scalar product $\langle \ , \ \rangle$. Let $J = \{1, \dots, m\} \subset \mathbb{N}$ resp. $J = \mathbb{N}$. If $J = \mathbb{N}$, then set $m = \infty$.

DEFINITION 5.1.17. *A system $\{f_k\}_{k \in J} \subset H$ is called* ORTHONORMAL *if for all $k, l \in J$*

$$\langle f_k, f_l \rangle = \delta_{kl} \ .$$

It is called an ORTHONORMAL BASIS OF H *if for all $f \in H$ there is $(a_n)_{n \in J} \subset \mathcal{A}$ such that $f = \sum_{n=1}^{m} f_n a_n$.*

Since $a_n = \langle f_n, f \rangle$, the coefficients are uniquely defined by the system.

PROPOSITION 5.1.18. *Let $\{f_k\}_{k \in J}$ be an orthonormal system in H whose span is dense in H. Then it is an orthonormal basis of H and the map*

$$f \mapsto (\langle f_n, f \rangle)_{n \in J}$$

is an isomorphism from H to $\mathcal{A}^m$ if $m < \infty$ and to $l^2(\mathcal{A})$ else.

PROOF. Let P_n be the orthogonal projection onto the span of the first n vectors of the system $\{f_k\}_{k\in J}$. On $\operatorname{span}_{\mathcal{A}}\{f_k \mid k \in J\}$ the projection P_n converges strongly to the identity for $n \to \infty$. Since $\|P_n\| = 1$ for all $n \in \mathbb{N}$, it follows that P_n converges strongly to the identity on H. $\qquad\square$

LEMMA 5.1.19. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}}U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators with $T_i \in B(U_i)$ and assume that there is $c \in \mathbb{R}$ such that $\|T_i\| \leq c$ for all $i \in \mathbb{N}$.*
Then the closure T of the operator $\oplus_{i\in\mathbb{N}}T_i$ is in $B(H)$ and $\|T\| \leq c$.

PROOF. The spectral radius of an operator $A \in B(H)$ is denoted by $r(A)$.
Write $T(n)$ for the restriction of T on $\oplus_{i=1}^n U_i$. Then for all $n \in \mathbb{N}$

$$\|T(n)\|^2 = r(T(n)^*T(n)) = \max_{1\leq i\leq n} r(T_i^*T_i) = \max_{1\leq i\leq n} \|T_i^*T_i\| \leq c^2 \ .$$

For $v \in \operatorname{dom} T$ there is $n \in \mathbb{N}$ such that $v \in \oplus_{i=1}^n U_i$. Then $Tv = T(n)v$ and thus

$$\|Tv\| = \|T(n)v\| \leq c\|v\| \ .$$

It follows that the closure of $\oplus_{i\in\mathbb{N}}T_i$ is a bounded operator on H. The adjoint is given by the closure of $\oplus_{i\in\mathbb{N}}T_i^*$. $\qquad\square$

COROLLARY 5.1.20. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of pairwise orthogonal closed subspaces of H such that $\oplus_{i\in\mathbb{N}}U_i$ is dense in H. Let $\{T_i\}_{i\in\mathbb{N}}$ be a family of operators such that $T_i^{-1} \in B(U_i)$ and assume that there is $c \in \mathbb{R}$ such that $\|T_i^{-1}\| \leq c$ for all $i \in \mathbb{N}$. Then the closure T of the map $\oplus_{i\in\mathbb{N}}T_i$ is invertible with inverse in $B(H)$.*

PROOF. The operator $\oplus_{i\in\mathbb{N}}T_i^{-1}$ is inverse to $\oplus_{i\in\mathbb{N}}T_i$. It fulfills the conditions of the previous lemma, hence its closure is a bounded operator on H. It is the inverse of the closure of T. $\qquad\square$

PROPOSITION 5.1.21. *Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of $l^2(\mathcal{A})$. Let M be a closed and N a projective submodule of $l^2(\mathcal{A})$ such that $l^2(\mathcal{A}) = M \oplus N$. Let P be the projection onto N along M and let P_n be the orthogonal projection onto $L_n := \operatorname{span}_{\mathcal{A}}\{e_i \mid i = 1, \ldots, n\}$. Assume that P is adjointable.*
For $n \in \mathbb{N}$ such that

$$\|P(1 - P_n)\| \leq \frac{1}{2}$$

it holds:

   (i) *The $\mathcal{A}$-module $N' := P_n(N)$ is projective and the maps*

$$P_n : N \to N' \text{ and } P : N' \to N$$

   are isomorphisms.
   (ii) *$l^2(\mathcal{A}) = M \oplus N'$.*

Note that for $n \in \mathbb{N}$ large enough $\|P(1 - P_n)\| \leq \frac{1}{2}$ since P is a compact operator.

PROOF. From $\|P(1 - P_n)\| \leq \frac{1}{2}$ it follows that

$$\|1_N - (PP_n)|_N\| \leq \frac{1}{2} \ .$$

Hence $(PP_n)|_N : N \to N$ is invertible.

The module $N' := P_n(N)$ is closed and finitely generated, thus projective. Furthermore, since $(PP_n)|_N$ is an isomorphism, the maps

$$P_n : N \to N' \text{ and } P : N' \to N$$

are isomorphisms as well.

It remains to show $N' \oplus M = l^2(\mathcal{A})$.

The intersection $N' \cap M$ is trivial: If $x \in N' \cap M$, then $Px = 0$, hence, as $P : N' \to N$ is an isomorphism, $x = 0$.

Let now $x \in l^2(\mathcal{A})$. Since $PP_n : N \to N$ is invertible, there is $y \in N$ such that $Px = PP_n y$.

Then

$$
\begin{aligned}
x &= (1-P)x + Px \\
  &= (1-P)x + PP_n y \\
  &= (1-P)x + P_n y - (1-P)P_n y \\
  &= (1-P)(x - P_n y) + P_n y \; .
\end{aligned}
$$

Since $(1-P)(x - P_n y) \in M$ and $P_n y \in N'$, it follows that $x \in M \oplus N'$. $\qquad\square$

5.2. Operators on spaces of vector valued functions

5.2.1. Vector valued functions and tensor products. Let V be a Fréchet space.

In the following we are interested in spaces $\Gamma(M, V)$ where $\Gamma = C, C^\infty, C^k, L^p$ and M is endowed with the appropriate structure. It is desirable to have an isomorphism $\Gamma(M, \mathbb{C}) \otimes V \cong \Gamma(M, V)$ extending the inclusion $\Gamma(M, \mathbb{C}) \odot V \hookrightarrow \Gamma(M, V)$ because this ensures that every bounded operator on $\Gamma(M, \mathbb{C})$ extends to a bounded operator on $\Gamma(M, V)$ (at least if the tensor product is an ε- or π-tensor product). Examples where such an isomorphism exists are listed below whereas the example $\Gamma = L^2$ where in general such an isomorphism cannot be found is studied in detail in the following sections.

Proofs can be found in [**Tr**]. If not specified the functions are assumed to be complex valued.

- Let M be a compact topological space. Then

$$C(M) \otimes_\varepsilon V \cong C(M, V) \; .$$

  For compact spaces M, N

$$C(M \times N) \cong C(M) \otimes_\varepsilon C(N) \; .$$

- Let $U \subset \mathbb{R}^n$ be open and precompact. For all $m \in \mathbb{N}_0$

$$C_0^m(U) \otimes_\varepsilon V \cong C_0^m(U, V) \; .$$

- Let M be a closed smooth manifold. For all $m \in \mathbb{N}_0$

$$C^m(M) \otimes_\varepsilon V \cong C^m(M, V) \; .$$

- Let $U \subset \mathbb{R}^n$ be open and precompact. Then $C_0^\infty(U)$ is nuclear, in particular

$$C_0^\infty(U) \otimes_\pi V \cong C_0^\infty(U) \otimes_\varepsilon V \cong C_0^\infty(U, V) \; .$$

- Let M be a closed smooth manifold. Then $C^\infty(M)$ is nuclear, in particular
$$C^\infty(M) \otimes_\pi V \cong C^\infty(M) \otimes_\varepsilon V \cong C^\infty(M, V) .$$
For closed smooth manifolds M, N
$$C^\infty(M \times N) \cong C^\infty(M) \otimes C^\infty(N) .$$

- The space of Schwartz functions $\mathcal{S}(\mathbb{R})$ is nuclear, in particular
$$\mathcal{S}(\mathbb{R}) \otimes_\pi V \cong \mathcal{S}(\mathbb{R}) \otimes_\varepsilon V \cong \mathcal{S}(\mathbb{R}, V) .$$

The isomorphisms are given by the unique extension of the inclusion of the algebraic tensor product.

5.2.2. L^2-spaces and integral operators. Let E be a Banach space with norm $|\cdot|$. Let $\operatorname{End} E$ be the Banach algebra of bounded operators on E. We denote the operator norm on $\operatorname{End} E$ by $|\cdot|$ as well.

DEFINITION 5.2.1. *Let M be a measure space and $p \in \mathbb{N}$.*
The Banach space $L^p(M, E)$ is defined to be the completion of the quotient of the space of simple E-valued functions on M by the subspace of functions vanishing almost everywhere with respect to the norm
$$\|f\|_{L^p} := \left(\int_M |f(x)|^p dx \right)^{\frac{1}{p}} .$$

In order to avoid confusion we make the following convention: If $E = \mathcal{A}^n$ for a C^*-algebra $\mathcal{A}$, then $L^2(M, E)$ denotes the Hilbert $\mathcal{A}$-module defined in §5.1.1 and not the space just defined. In general these spaces do not coincide.

LEMMA 5.2.2. *Let M_1, M_2 be σ-finite measure spaces. Then the map*
$$L^2(M_1 \times M_2, E) \to L^2(M_1, L^2(M_2, E)), \ f \mapsto (x \mapsto f(x, \cdot))$$
is an isometric isomorphism.

PROOF. The lemma follows from Fubini. $\square$

PROPOSITION 5.2.3. *Let M be a measure space.*
Let $k : M \times M \to \operatorname{End} E$ be a measurable function such that the integral kernel $|k(x, y)|$ defines a bounded operator $|K|$ on $L^2(M)$ with norm $\||K|\|$. Then k defines a bounded operator on $L^2(M, E)$ with norm less than or equal to $\||K|\|$.

PROOF. For a simple function $f : M \to E$
$$\| \int_M k(\cdot, y) f(y) \ dy\|_{L^2} \le \| \int_M |k(\cdot, y)||f(y)| \ dy\|_{L^2} \le \||K|\| \ \|f\|_{L^2}.$$
$\square$

COROLLARY 5.2.4. *Let M be a measure space.*
There is a norm-decreasing map
$$L^2(M \times M, \operatorname{End} E) \to B(L^2(M, E), L^2(M, E))$$
$$k \mapsto \left(f \mapsto Kf := \int_M k(\cdot, y) f(y) \ dy \right) .$$

COROLLARY 5.2.5. *The convolution induces a continuous map*
$$L^1(\mathbb{R}^n, \operatorname{End} E) \to B(L^2(\mathbb{R}^n, E)), \ f \mapsto (g \mapsto f * g) .$$

PROOF. The convolution with $f \in L^1(\mathbb{R}^n, \operatorname{End} E)$ is an integral operator with integral kernel $f(x-y)$. Since $|f| \in L^1(\mathbb{R}^n)$ for $f \in L^1(\mathbb{R}^n, \operatorname{End} E)$, the convolution with $|f|$ is bounded on $L^2(\mathbb{R}^n)$. Then the assertion follows from the previous proposition. $\square$

LEMMA 5.2.6. *For any $f \in L^p(\mathbb{R}^n, E)$ the translation map*

$$\tau f : \mathbb{R}^n \to L^p(\mathbb{R}^n, E), \ y \mapsto \tau_y f \ ,$$

defined by

$$\tau_y f(x) := f(x-y) \ ,$$

is continuous.

PROOF. The proof is analogous to the case $E = \mathbb{C}$, see [**Co**], Ch. VII, Prop. 9.2. $\square$

5.2.3. Adjointable operators on Banach spaces. Let $\mathcal{B}$ be an involutive Banach algebra with unit. In this section all operators are assumed to be right $\mathcal{B}$-module maps. Let E be a Banach right $\mathcal{B}$-module with norm $|\cdot|$.

DEFINITION 5.2.7. *A $\mathcal{B}$-VALUED NON-DEGENERATED PRODUCT ON E is a map $\langle \ , \ \rangle : E \times E \to \mathcal{B}$ that is $\mathbb{C}$-linear in the second variable and has the following properties:*

(1) $\langle v, wb \rangle = \langle v, w \rangle b$ *for all $v, w \in E$, $b \in \mathcal{B}$,*
(2) $\langle v, w \rangle = \langle w, v \rangle^*$ *for all $v, w \in E$,*
(3) *if $\langle v, w \rangle = 0$ for all $w \in E$, then $v = 0$,*
(4) *there is $C > 0$ such that $|\langle v, w \rangle| \leq C|v|_E \ |w|_E$ for all $v, w \in E$.*

Let E be endowed with a $\mathcal{B}$-valued non-degenerated product $\langle \ , \ \rangle$.

DEFINITION 5.2.8. *A bounded operator $T : E \to E$ is called ADJOINTABLE if there is a map $T^* : E \to E$ satisfying*

$$\langle v, Tw \rangle = \langle T^*v, w \rangle$$

for all $v, w \in E$.

LEMMA 5.2.9. *Let $T : E \to E$ be adjointable.*

(1) *The adjoint T^* is unique.*
(2) *T^* is a right $\mathcal{B}$-module map.*
(3) *T^* is bounded.*
(4) *$T^{**} = T$.*
(5) *$(ST)^* = T^*S^*$.*

PROOF. (1) Let T^* be an adjoint of T and $\Gamma(T^*)$ its graph. Then

$$\Gamma(T^*) \subset G := \{(x,y) \in E \times E \mid \langle y, w \rangle + \langle -x, Tw \rangle = 0 \ \forall w \in E\} \ .$$

Let $v \in E$. There is a unique $v_1 \in E$ with $(v, v_1) \in G$ since from

$$\langle v_1, w \rangle = \langle v, Tw \rangle = \langle v_2, w \rangle \ \forall w \in E$$

it follows $\langle v_1 - v_2, w \rangle = 0$ for all $w \in E$ and therefore $v_1 = v_2$. This shows $\Gamma(T^*) = G$.

(2) If $(x,y), (v,w) \in \Gamma(T^*)$ and $b \in \mathcal{B}$, then $(xb + v, yb + w) \in \Gamma(T^*)$ by the proof of (1).

(3) Since $\Gamma(T^*)$ is closed, the operator T^* is bounded.

(4) From (1) it follows that $\Gamma(T^{**}) = \Gamma(T)$.

(5) $\langle (ST)^* v, w \rangle = \langle v, STw \rangle = \langle S^* v, Tw \rangle = \langle T^* S^* v, w \rangle.$ $\qquad\square$

LEMMA 5.2.10. *Let T be a densely defined operator on E. If there exists a densely defined operator S, called* FORMAL ADJOINT *of T, such that*

$$\langle f, Tg \rangle = \langle Sf, g \rangle$$

for all $f \in \operatorname{dom} S$, $g \in \operatorname{dom} T$, then T is closable.

PROOF. The set

$$\{(x, y) \in E \oplus E \mid \langle f, y \rangle - \langle Sf, x \rangle = 0 \ \forall f \in \operatorname{dom} S\}$$

is the graph of a closed extension of T. $\qquad\square$

The standard $\mathcal{B}$-valued non-degenerated product on $\mathcal{B}^n$ is given by

$$\langle v, w \rangle := \sum_{i=1}^{n} v_i^* w_i \ .$$

Since the endomorphism set of $\mathcal{B}^n$ can be identified with $M_n(\mathcal{B})$, all elements of $\operatorname{End}(\mathcal{B})$ are adjointable and taking the adjoint is a bounded linear map.

If M is a measure space, the standard $\mathcal{B}$-valued non-degenerated product on $L^2(M, \mathcal{B}^n)$ is defined by

$$\langle f, g \rangle_{L^2} := \int_M \langle f(x), g(x) \rangle dx \ .$$

We check condition (3) of Def. 5.2.7:

Let $\mathcal{B}'$ be the topological dual of $\mathcal{B}$ endowed with the weak topology.

We use that every $\lambda \in \mathcal{B}'$ induces a map $\lambda : \mathcal{B}^n \to \mathbb{C}^n$ by componentwise application.

If $f \in L^2(M, \mathcal{B}^n)$ with $\langle f, g \rangle_{L^2} = 0$ for all $g \in L^2(M, \mathcal{B}^n)$, then in particular for all $g \in L^2(M, \mathbb{C}^n)$ and $\lambda \in \mathcal{B}'$

$$\int_M \lambda(f(x)^*) g(x) dx = 0 \ ,$$

hence $\lambda(f(x)^*)$ vanishes almost everywhere. Since $\mathcal{B}'$ is separable, it follows $f = 0$ in $L^2(M, \mathcal{B}^n)$.

5.2.4. Hilbert-Schmidt operators. Let the notation be as in the previous section. Assume that M is a σ-finite measure space.

LEMMA 5.2.11. *Let $k \in L^2(M \times M, M_n(\mathcal{B}))$ and let K be the corresponding integral operator on $L^2(M, \mathcal{B}^n)$. Then k is uniquely defined by K.*

PROOF. It is enough to show that k vanishes in $L^2(M \times M, M_n(\mathcal{B}))$ if $K = 0$.

Applying $\lambda \in \mathcal{B}'$ componentwise yields maps $\lambda : M_n(\mathcal{B}) \to M_n(\mathbb{C})$ and $\lambda : \mathcal{B}^n \to \mathbb{C}^n$.

For $f \in L^2(M, \mathbb{C}^n)$ we have that almost everywhere

$$\lambda \left(\int_M k(x, y) f(y) \ dy \right) = \int_M \lambda(k(x, y)) f(y) \ dy = 0 \ .$$

It follows that $\lambda(k(x, y)) = 0$ in $L^2(M \times M, M_n(\mathbb{C}))$. Since $\mathcal{B}'$ is separable, the set of all $(x, y) \in M \times M$ such that there is $\lambda \in \mathcal{B}'$ with $\lambda(k(x, y)) \neq 0$ has measure zero. On the complement of this set k vanishes. $\qquad\square$

DEFINITION 5.2.12. *A* HILBERT-SCHMIDT OPERATOR ON $L^2(M, \mathcal{B}^n)$ *is an integral operator with integral kernel in* $L^2(M \times M, M_n(\mathcal{B}))$. *Let A be a Hilbert-Schmidt operator on* $L^2(M, \mathcal{B}^n)$ *with integral kernel* $k_A \in L^2(M \times M, M_n(\mathcal{B}))$. *We define*

$$\|A\|_{HS} := \|k_A\| \ ,$$

where the norm on the right hand side is taken in $L^2(M \times M, M_n(\mathcal{B}))$.

The normed space of Hilbert-Schmidt operators on $L^2(M, \mathcal{B}^n)$ *is denoted by* $HS(L^2(M, \mathcal{B}^n))$.

Note that $HS(L^2(M, \mathcal{B}^n))$ is a Banach algebra and that the inclusion

$$HS(L^2(M, \mathcal{B}^n)) \to B(L^2(M, \mathcal{B}^n))$$

is bounded. Prop. 5.2.13 below shows that $HS(L^2(M, \mathcal{B}^n))$ is a left $B(L^2(M, \mathcal{B}^n))$-module.

All operators in $HS(L^2(M, \mathcal{B}^n))$ are adjointable. The integral kernels of $A \in HS(L^2(M, \mathcal{B}^n))$ and A^* are related by $k_{A^*}(x, y) = k_A(y, x)^*$. It follows that taking the adjoint is a bounded map on $HS(L^2(M, \mathcal{B}^n))$.

PROPOSITION 5.2.13. *Let* $A \in B(L^2(M, \mathcal{B}^n))$, $K \in HS(L^2(M, \mathcal{B}^n))$.

(1) *Then* $AK \in HS(L^2(M, \mathcal{B}^n))$. *Furthermore there is* $C > 0$, *independent of A and K, such that*

$$\|AK\|_{HS} \le C\|A\|\|K\|_{HS} \ .$$

(2) *If A is adjointable, then* $KA \in HS(L^2(M, \mathcal{B}^n))$. *Furthermore there is* $C > 0$, *independent of A and K, with*

$$\|KA\|_{HS} \le C\|A^*\|\|K\|_{HS} \ .$$

PROOF. (1) There is an isomorphism

$$L^2(M \times M, M_n(\mathcal{B})) \cong L^2(M \times M, \mathcal{B}^n)^n$$

that is equivariant with respect to the left $M_n(\mathcal{B})$-action on both spaces. Furthermore the map

$$L^2(M \times M, \mathcal{B}^n) \to L^2(M, L^2(M, \mathcal{B}^n)), \ k \mapsto (y \mapsto k(\cdot, y))$$

is an isomorphism by Lemma 5.2.2. The operator A induces a bounded map on $L^2(M, L^2(M, \mathcal{B}^n))$, namely $k \mapsto (y \mapsto Ak(\cdot, y))$, clearly its norm is less than or equal to the norm of A on $L^2(M, \mathcal{B}^n)$.

(2) The map $K \mapsto KA$ is a composition of the following maps on $HS(L^2(M, \mathcal{B}^n))$:

$$K \overset{*}{\mapsto} K^* \overset{A^*}{\mapsto} A^*K^* = (KA)^* \overset{*}{\mapsto} KA \ .$$

By (1) and the fact that taking the adjoint is bounded on $HS(L^2(M, \mathcal{B}^n))$ these maps are bounded. □

DEFINITION 5.2.14.   (1) *Let* $\langle \ , \ \rangle : \mathcal{B}^n \times \mathcal{B}^n \to \mathcal{B}$ *be the standard* $\mathcal{B}$-*valued non-degenerated product. For* $e \in \mathcal{B}^n$ *define the map*

$$e^* : \mathcal{B}^n \to \mathcal{B}, \ v \mapsto \langle e, v \rangle \ .$$

(2) *An integral operator A on* $L^2(M, \mathcal{B}^n)$ *is called* FINITE *if there is* $k \in \mathbb{N}$ *and there are functions* $f_j, h_j \in L^2(M, \mathcal{B}^n)$, $j = 1 \ldots k$, *such that*

$$k_A(x, y) = \sum_{j=1}^{k} f_j(x) h_j(y)^* \ .$$

5.2.5. Trace class operators. Let M be a σ-finite Borel space. Assume that there is a uniformly bounded sequence $(K_m)_{m \in \mathbb{N}} \subset B(L^2(M, \mathcal{B}^n))$ converging strongly to the identity such that each K_m is an integral operator with continuous compactly supported integral kernel.

This condition is fulfilled for example if M is a complete Riemannian manifold: Using Prop. 5.2.3 and the fact that the heat kernel $k_t(x, y)$ of the scalar Laplacian Δ is positive, one can deduce that $k_t(x, y)$ defines a strongly continuous semigroup $e^{-t\Delta}$ on $L^2(M, \mathcal{B}^n)$. Then $K_m := \phi_m e^{-\frac{1}{m}\Delta}\phi_m$ is an appropriate sequence, where $(\phi_m)_{m \in \mathbb{N}}$ is a sequence in $C_c(M)$ that converges to the identity on compact subsets of M.

The composition of operators induces a continuous map

$$\mu : HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n)) \to B(L^2(M, \mathcal{B}^n)) \ .$$

DEFINITION 5.2.15. *A* TRACE CLASS OPERATOR *on* $L^2(M, \mathcal{B}^n)$ *is an element in the range of* μ. *The space of trace class operators is denoted by* $\mathrm{Tr}(L^2(M, \mathcal{B}^n))$. *We identify it with*

$$\big(HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n))\big)/\operatorname{Ker}\mu$$

and endow it with the quotient norm.

It follows from Prop. 5.2.13 that $\mathrm{Tr}(L^2(M, \mathcal{B}^n))$ is a left $B(L^2(M, \mathcal{B}^n))$-module and that there is a well-defined action of adjointable operators from the right.

PROPOSITION 5.2.16. *The map*

$$R : HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n)) \to L^1(M, M_n(\mathcal{B})) \ ,$$

$$(A, B) \to (x \mapsto \int_M k_A(x, y)k_B(y, x) \ dy)$$

descends to a continuous map

$$\overline{R} : \mathrm{Tr}(L^2(M, \mathcal{B}^n)) \to L^1(M, M_n(\mathcal{B})) \ .$$

PROOF. Note that R is continuous, hence all we have to show is that $\overline{R}$ is well-defined. Since the sequence K_m is uniformly bounded, the map $F \mapsto K_m F K_m$ on $HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n))$ converges strongly to the identity. The assertion follows if we can show that $\mu(F) = 0$ implies $R(K_m F K_m) = 0$.

The map $F \mapsto K_m F K_m$ is continuous as a map from $HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n))$ to $C_c(M, L^2(M, M_n(\mathcal{B}))) \otimes_\pi C_c(M, L^2(M, M_n(\mathcal{B})))$. The composition with the continuous map $C_c(M, L^2(M, M_n(\mathcal{B}))) \otimes_\pi C_c(M, L^2(M, M_n(\mathcal{B}))) \to C_c(M \times M, M_n(\mathcal{B}))$ induced by the product

$$L^2(M, M_n(\mathcal{B})) \times L^2(M, M_n(\mathcal{B})) \to M_n(\mathcal{B}), \ (f, g) \mapsto \int_M f(x)g(x) \ dx$$

agrees with the map that sends F to the continuous integral kernel $k_{\mu(K_m F K_m)}$ of $\mu(K_m F K_m)$. We compose further with the restriction to the diagonal and get a continuous map

$$HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n)) \to C_c(M, M_n(\mathcal{B})) \subset L^1(M, M_n(\mathcal{B})) \ ,$$

which agrees with the map $F \mapsto R(K_m F K_m)$. This shows that the map $F \mapsto R(K_m F K_m)$ factors through the map

$$HS(L^2(M, \mathcal{B}^n)) \otimes_\pi HS(L^2(M, \mathcal{B}^n)) \to C_c(M \times M, M_n(\mathcal{B})), \ F \mapsto k_{\mu(K_m F K_m)} \ .$$

Now $\mu(F) = 0$ implies $\mu(K_m F K_m) = 0$, hence $k_{\mu(K_m F K_m)} = 0$ by the uniqueness of the integral kernel in $C_c(M \times M, M_n(\mathcal{B}))$. $\qquad\square$

Let
$$\mathrm{tr} : M_n(\mathcal{B}) \to \mathcal{B}/\overline{[\mathcal{B}, \mathcal{B}]}$$
be the trace defined, as usual, by adding up the diagonal elements and let
$$\mathrm{Tr}(T) := \int_M \mathrm{tr}\ \overline{R}(T)(x)\ dx$$
for $T \in \mathrm{Tr}(L^2(M, M_n(\mathcal{B})))$. Then for Hilbert-Schmidt operators A, B
$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)\ .$$

From the fact that any trace class operator can be approximated by finite sums of products of Hilbert-Schmidt operators it follows that the equation holds also for $A \in \mathrm{Tr}(L^2(M, \mathcal{B}^n))$ and B any adjointable operator.

5.2.6. Pseudodifferential operators. Let E be a Banach space.

Let U be an open precompact subset of $\mathbb{R}^n$. Recall the notion of a symbol of order m on U:

DEFINITION 5.2.17. *A function $a \in C^\infty(U \times \mathbb{R}^n, M_l(\mathbb{C}))$ is called a* SYMBOL OF ORDER $m \in \mathbb{R}$ *if it is compactly supported in the first variable and if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ the expressions*
$$\sup_{x \in U,\ \xi \in \mathbb{R}^n} (1 + |\xi|)^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|$$
are finite.

These are seminorms on the space $S^m(U, M_l(\mathbb{C}))$ of symbols of order m on U, which turn $S^m(U, M_l(\mathbb{C}))$ into a Fréchet space.

In order to simplify formula involving Fourier transform it is convenient to rescale the Lebesgue measure on $\mathbb{R}^n$ by setting $d'x := (2\pi)^{-\frac{n}{2}} dx$.

We consider $L^2(U, E^l)$ as a subspace of $L^2(\mathbb{R}^n, E^l)$ in the following.

The Fourier transform is bounded from $L^1(\mathbb{R}^n, E)$ to $C_0(\mathbb{R}^n, E)$.

A symbol $a \in S^m(U, M_l(\mathbb{C}))$ defines a continuous operator
$$\mathrm{Op}(a) : C_c^\infty(U, E^l) \to C_c^\infty(U, E^l),\ (\mathrm{Op}(a)f)(x) = \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi)\ d'\xi$$
called a pseudodifferential operator of order m.

Due to the fact that the Fourier transform is in general not continuous on $L^2(\mathbb{R}^n, E^l)$, the continuity properties of pseudodifferential operators acting on vector valued functions are in general weaker that those one gets for $E = \mathbb{C}$. They are still weaker if we allow for symbols with values in $\mathrm{End}\, E$.

LEMMA 5.2.18.   (1) *For $m < -\frac{n}{2}$ and $a \in S^m(U, M_l(\mathbb{C}))$ the operator $\mathrm{Op}(a)$ extends to a bounded operator on $L^2(U, E^l)$ and the map*
$$\mathrm{Op} : S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l))$$
is continuous.

(2) *Let $m < -\frac{n}{2}$ and $\nu, k \in \mathbb{N}_0$ with $k < -\frac{n}{2} - m$. Then for $a \in S^m(U, M_l(\mathbb{C}))$
the operators*

$$\mathrm{Op}(a) : C_0^\nu(U, E^l) \to C_0^{\nu+k}(U, E^l)$$

and

$$\mathrm{Op}(a) : L^2(U, E^l) \to C_0^k(U, E^l)$$

are continuous.

PROOF. (1) Let $m < -\frac{n}{2}$.
The Fourier transform in ξ induces a bounded map

$$S^m(U, M_l(\mathbb{C})) \to C_c^\infty(U, L^2(\mathbb{R}^n, M_l(\mathbb{C}))), \ a \mapsto (x \mapsto \hat{a}(x, \cdot)) \ .$$

If $a \in S^m(U, M_l(\mathbb{C}))$, we extend $\mathrm{Op}(a)$ to $L^2(U, E^l)$ by

$$(\mathrm{Op}(a)f)(x) := \int_{\mathbb{R}^n} \hat{a}(x, z) f(-x - z) d'z \ .$$

The map

$$S^m(U, M_l(\mathbb{C})) \to B(L^2(U, E^l)), \ a \mapsto \mathrm{Op}(a)$$

is well-defined and continuous since

$$\mathbb{R}^n \to L^2(\mathbb{R}^n, E^l), \ x \mapsto (z \mapsto f(-x - z))$$

is continuous by Lemma 5.2.6, hence $\mathrm{Op}(a)f \in C_0(U, E^l)$ and

$$\| \mathrm{Op}(a)f \|_{C_0} \leq \sup_{x \in U} \| \hat{a}(x, \cdot) \|_{L^2} \| f \|_{L^2} \ .$$

(2) First let $m < -\frac{n}{2}$ and $k = 0$.
If $f \in C_0^\nu(U, E^l)$, then $x \mapsto (z \mapsto f(-x - z))$ is in $C_0^\nu(\mathbb{R}^n, L^2(\mathbb{R}^n, E^l))$.
It follows as above that $\mathrm{Op}(a)f \in C_0^\nu(U, E^l)$ and

$$\| \mathrm{Op}(a)f \|_{C^\nu} \leq C \sup_{|\alpha| \leq \nu} \sup_{x \in U} \| \partial_x^\alpha \hat{a}(x, \cdot) \|_{L^2} \| f \|_{C^\nu} \ .$$

Now assume that the assertion holds for $k - 1$ and all $a \in S^m(U, M_l(\mathbb{C}))$ with
$m < -\frac{n}{2} - k + 1$.
We prove the assertion for k and $a \in S^m(U, M_l(\mathbb{C}))$ with $m < -\frac{n}{2} - k$:
If $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, then the map

$$f \mapsto \partial^\alpha(\mathrm{Op}(a)f)$$

is a pseudodifferential operator of degree $m + 1$. By induction it is a bounded
operator from $C^\nu(U, E^l)$ to $C^{\nu+k-1}(U, E^l)$. It follows that $\mathrm{Op}(a)$ is continuous
from $C^\nu(U, E^l)$ to $C^{\nu+k}(U, E^l)$.

An analogous induction argument shows that $\mathrm{Op}(a)$ is continuous from $L^2(U, E^l)$
to $C_0^k(U, E^l)$. For $k = 0$ this was proved in (1). $\qquad\square$

5.3. Projective systems and function spaces

The projective systems $(\mathcal{A}_i)_{i \in \mathbb{N}_0}$ and $(\hat{\Omega}_{\leq \mu} \mathcal{A}_i)_{i, \mu \in \mathbb{N}_0}$ from §1.3.3 induce pro-
jective systems of spaces $\left(L^2(M, \mathcal{A}_i^l)\right)_{i \in \mathbb{N}_0}$ and $\left(L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)\right)_{i, \mu \in \mathbb{N}_0}$:

Recall the convention fixed in §5.2.2: The space $L^2(M, \mathcal{A}^l)$ is the Hilbert $\mathcal{A}$-
module defined in §5.1.1. For $\mu \in \mathbb{N}_0$ and $i \in \mathbb{N}$ the space $L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$ was
defined in §5.2.2.

In the following we investigate the behavior of some particular classes of oper-
ators on $L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$ under the projective limit.

5.3.1. Integral operators. Hilbert-Schmidt operators on $L^2(M, \mathcal{A}_i^l)$ have the property that they extend to bounded operators on $L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_j)^l)$ for all $j \in \mathbb{N}_0$ with $j \leq i$ and all $\mu \in \mathbb{N}_0$. We investigate how the spectrum depends on j, μ.

We extend a method developed by Lott ([**Lo3**], §6.1.) for closed manifolds to certain non-compact manifolds with boundary (in particular for the manifold defined in §1.1).

Let $[0,1]^n$ be endowed with a measure of the form hdx where h is a positive continuous function on $[0,1]^n$ and dx is the Lebesgue measure.

In the proof of the following proposition we use that there exists a Schauder basis of $C([0,1]^n)$ that is orthonormal in $L^2([0,1]^n)$ (here and in the following we understand $L^2([0,1]^n)$ with respect to hdx). For $h = 1$ a Franklin system [**Se**] yields such a basis $\{f_n\}_{n \in \mathbb{N}}$, then for general h the system $\{h^{-\frac{1}{2}} f_n\}_{n \in \mathbb{N}}$ is one.

The proposition still holds true if $[0,1]^n$ is replaced by a compact Borel space for which such a basis exists.

PROPOSITION 5.3.1.   (1) *Let $[0,1]^n$ be endowed with a measure hdx as above. Let $k \in C([0,1]^n \times [0,1]^n, M_l(\mathcal{A}_i))$ and let K be the corresponding integral operator. Assume that $1 - K$ is invertible in $B(L^2([0,1]^n, \mathcal{A}^l))$. Then $1 - K : L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l) \to L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$ is invertible.*
    (2) *Let M be a Riemannian manifold of dimension n, possibly with boundary. Suppose that there is a covering $\{K_m\}_{m \in \mathbb{N}}$ of M with K_m compact, $K_m \subset K_{m+1}$ and such that K_m is diffeomorphic to $[0,1]^n$ for every $m \in \mathbb{N}$. Let $k \in L^2(M \times M, M_l(\mathcal{A}_i)) \cap C(M \times M, M_l(\mathcal{A}_i))$ and assume furthermore that $x \mapsto k(x, \cdot)$ and $y \mapsto k(\cdot, y)$ are in $C(M, L^2(M, M_l(\mathcal{A}_i)))$.*
    If $1 - K$ is invertible in $B(L^2(M, \mathcal{A}^l))$, then $1 - K$ is invertible in $B(L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l))$.

PROOF. (1) Choose a basis of $C([0,1]^n)$ which is orthonormal with respect to hdx and let P_N denote the projection onto the first N basis vectors. The integral kernel of P_N is in $L^2([0,1]^n \times [0,1]^n)$, hence P_N acts continuously on $L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$.

We decompose $L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$ into the direct sum

$$P_N L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l) \oplus (1 - P_N) L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$$

and write

$$1 - K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition.

If we find N such that d is invertible on $(1 - P_N) L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$ and prove that then $a - bd^{-1}c$ is invertible on $P_N L^2([0,1]^n, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)$, we can conclude that $(1 - K)$ is invertible by the equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.$$

First we show that

$$d = (1 - P_N)(1 - K)(1 - P_N)$$

is invertible for N big enough. By Prop. 5.2.13 the operator $(1 - P_N)K(1 - P_N)$ is a Hilbert-Schmidt operator. Its integral kernel is continuous.

For $N \to \infty$ the projections P_N converge strongly to the identity on $C([0,1]^n)$. Since

$$C([0,1]^n \times [0,1]^n, M_l(\mathcal{A}_i)) \cong C([0,1]^n) \otimes_\varepsilon C([0,1]^n) \otimes_\varepsilon M_l(\mathcal{A}_i) \ ,$$

there is N such that the norm of the integral kernel of $(1 - P_N)K(1 - P_N)$ is smaller than $\frac{1}{2}$ in $C([0,1]^n \times [0,1]^n, M_l(\mathcal{A}_i))$.

For that N the series

$$(1 - P_N) + \sum_{\nu=1}^{\infty}((1 - P_N)K(1 - P_N))^\nu$$

converges as a bounded operator on $(1 - P_N)L^2([0,1]^n, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ and inverts d. Hence $a - bd^{-1}c$ is well-defined.

Via the basis we identify $a - bd^{-1}c$ with an element of $M_{Nl}(\mathcal{A}_i)$. Since $1 - K$ is invertible on $L^2(M, \mathcal{A}^l)$, the matrix $a - bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A})$. By Prop. 1.3.4 it follows that $a - bd^{-1}c$ is invertible in $M_{Nl}(\mathcal{A}_i)$ as well.

(2) Let $m \in \mathbb{N}$ be such that in $L^2(M \times M, M_l(\mathcal{A}_i))$

$$\|(1 - 1_{K_m}(x))k(x,y)(1 - 1_{K_m}(y))\| \leq \frac{1}{2} \ .$$

Write

$$(1 - K) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition

$$L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) = 1_{K_m}L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \oplus (1 - 1_{K_m})L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l) \ .$$

By the choice of $m \in \mathbb{N}$ the entry $d = 1 - (1 - 1_{K_m})K(1 - 1_{K_m})$ is invertible on $(1 - 1_{K_m})L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$.

We prove that $a - bd^{-1}c$ is invertible on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ and then the assertion follows as in the proof of (1).

On $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$

$$a - bd^{-1}c = 1_{K_m} - (1_{K_m}K1_{K_m} + bd^{-1}c) \ .$$

The operator $1_{K_m}K1_{K_m} + bd^{-1}c$ is an integral operator on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with continuous integral kernel: The integral kernel of b is $1_{K_m}(x)k(x,y)(1 - 1_{K_m}(y))$ and $x \mapsto 1_{K_m}(x)k(x,\cdot)(1 - 1_{K_m})$ is in $C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. For the integral kernel $(1 - 1_{K_m}(x))k(x,y)1_{K_m}(y)$ of c we have $y \mapsto (1 - 1_{K_m})k(\cdot,y)1_{K_m}(y) \in C(K_m, L^2(M, M_l(\mathcal{A}_i)))$. It follows that $bd^{-1}c$ is an integral operator with continuous kernel on $K_m \times K_m$. Clearly the integral kernel of $1_{K_m}K1_{K_m}$ is continuous as well.

Since $a - bd^{-1}c$ is invertible on $L^2(K_m, \mathcal{A}^l)$ and since the measure on K_m pulled back by an orientation preserving diffeomorphism $[0,1]^n \to K_m$ is of the form hdx, we conclude by (1) that $a - bd^{-1}c$ is invertible on $L^2(K_m, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. $\square$

COROLLARY 5.3.2. *Let k be an integral kernel as in part (2) of the proposition and let K be the corresponding integral operator. Then for $\lambda \in \mathbb{C}^*$ the operator $K - \lambda$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ if $K - \lambda$ is invertible on $L^2(M, \mathcal{A}^l)$.*

PROOF. For $\lambda \in \mathbb{C} \setminus \{0\}$ the integral kernel k/λ fulfills the conditions of the lemma. Thus if $\lambda - K = \lambda(1 - K/\lambda)$ is invertible on $L^2(M, \mathcal{A}^l)$, then $\lambda - K$ is invertible on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ as well. $\square$

5.3.2. The Chern character.

DEFINITION 5.3.3. *Let V be a $\mathbb{Z}/2$-graded finite dimensional vector space. Let K be an integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $k : M \times M \to \mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$. Then we define $\mathrm{d}(K)$ to be the integral operator on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$ with integral kernel $\mathrm{d}(k(x,y))$. (The action of d on $\mathrm{End}(V) \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i$ was described in §1.3.2.)*

Note that if K is of degree n with respect to the $\mathbb{Z}/2$-grading on $L^2(M, V \otimes \hat{\Omega}_{\leq\mu}\mathcal{A}_i)$, then

$$\mathrm{d}(Kf) = \mathrm{d}(K)f + (-1)^n K \, \mathrm{d}\, f \ .$$

LEMMA 5.3.4. *Let M be a σ-finite measure space.*
Let P be a Hilbert-Schmidt operator on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ with integral kernel in $L^2(M \times M, M_l(\mathcal{A}_i))$ and assume that $P^2 = P$. Then

$$P \, \mathrm{d}\, P \, \mathrm{d}\, P = P(\mathrm{d}(P))^2 \ .$$

PROOF. As for matrices (see the beginning of the proof of Prop. 1.3.3) we have that $(\mathrm{d}\, P) = P(\mathrm{d}\, P) + (\mathrm{d}\, P)P$ and $P(\mathrm{d}\, P)P = 0$.
It follows that

$$
\begin{aligned}
P \, \mathrm{d}\, P \, \mathrm{d}\, P &= P(\mathrm{d}\, P)\, \mathrm{d}\, P = P(\mathrm{d}\, P)^2 P \\
&= P(\mathrm{d}\, P)(\mathrm{d}\, P) - P(\mathrm{d}\, P)P(\mathrm{d}\, P) \\
&= P(\mathrm{d}(P))^2.
\end{aligned}
$$

$\square$

LEMMA 5.3.5. *Let $P : [0,1] \to HS(L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l))$ be a differentiable path of Hilbert-Schmidt operators with integral kernels in $L^2(M \times M, M_l(\mathcal{A}_i))$, and assume that $P(t)^2 = P(t)$ for every $t \in [0,1]$.*
Then for $k \in \mathbb{N}_0$ in $\hat{\Omega}_{\leq\mu}\mathcal{A}_i/[\hat{\Omega}_{\leq\mu}\mathcal{A}_i, \hat{\Omega}_{\leq\mu}\mathcal{A}_i]$

$$\mathrm{Tr}\, P(1)(\mathrm{d}\, P(1))^{2k} - \mathrm{Tr}\, P(0)(\mathrm{d}\, P(0))^{2k}$$

is exact.

PROOF. As for matrices (see Prop. 1.3.3). $\square$

If T is a trace class operator on $L^2(M, (\hat{\Omega}_{\leq\mu}\mathcal{A}_i)^l)$ restricting to a trace class operator on $L^2(M, (\hat{\Omega}_{\leq\nu}\mathcal{A}_j)^l)$ for all $j, \nu \in \mathbb{N}$ with $j \geq i$ and $\nu \geq \mu$, then by taking the projective limit we can consider $\mathrm{Tr}(T)$ as in element in $\hat{\Omega}_*\mathcal{A}_\infty/[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty]_s$. In the next lemma we show that under certain conditions the formula for the Chern character can be generalized to projections which are Hilbert-Schmidt operators.

PROPOSITION 5.3.6. *Let M be a Riemannian manifold of dimension d, possibly with boundary. Suppose that there is a covering $\{K_m\}_{m\in\mathbb{N}}$ of M with K_m compact, $K_m \subset K_{m+1}$ and such that K_m is diffeomorphic to $[0,1]^d$ for all $m \in \mathbb{N}$.*
Let $P \in B(L^2(M, \mathcal{A}^l))$ be a projection onto a projective submodule of $L^2(M, \mathcal{A}^l)$. Assume further that for any $i \in \mathbb{N}$ it restricts to a bounded projection on $L^2(M, \mathcal{A}_i^l)$ and that $P(L^2(M, \mathcal{A}_i^l)) \subset C(M, \mathcal{A}_i^l)$. Let

$$\mathrm{Ran}_\infty P := \bigcap_{i\in\mathbb{N}} P(L^2(M, \mathcal{A}_i^l)) \ .$$

(1) *The projection P is a Hilbert-Schmidt operator with integral kernel of the form $\sum_{j=1}^{m} f_j(x) h_j(y)^*$ with $f_j, h_j \in \mathrm{Ran}_\infty P$.*
(2) *The intersection $\mathrm{Ran}_\infty P$ is a projective $\mathcal{A}_\infty$-module. The classes $[\mathrm{Ran}\, P] \in K_0(\mathcal{A})$ and $[\mathrm{Ran}_\infty P] \in K_0(\mathcal{A}_\infty)$ correspond to each other under the canonical isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\infty)$.*
(3) *In $H_*^{dR}(\mathcal{A}_\infty)$*

$$\mathrm{ch}[\mathrm{Ran}_\infty P] = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \mathrm{Tr}(P \, \mathrm{d}\, P)^{2n} \ .$$

PROOF. (1) Let $\{e_n\}_{n \in \mathbb{N}} \subset C_0^\infty(M, \mathbb{C}^l)$ be an orthonormal basis of $L^2(M, \mathbb{C}^l)$.

The orthogonal projection P_n onto the span of the first n basis vectors is a Hilbert-Schmidt operator with integral kernel in $C_0^\infty(M \times M, M_l(\mathbb{C}))$.

In particular $P_n \in B(L^2(M, \mathcal{A}_i^l))$ for any $i \in \mathbb{N}$.

First we consider the situation on $L^2(M, \mathcal{A}^l)$:

Since P is compact, there is $n \in \mathbb{N}$ such that on $L^2(M, \mathcal{A}^l)$

$$\|P(P_n - 1)\| \leq \frac{1}{2} \ .$$

Then by Prop. 5.1.21 the map $P P_n P : \mathrm{Ran}\, P \to \mathrm{Ran}\, P$ is an isomorphism.

It follows that $\mathrm{Ker}\, P P_n P = (\mathrm{Ran}\, P)^\perp = \mathrm{Ker}\, P$ and therefore

$$P = 1 - P_{\mathrm{Ker}\, P P_n P} \ .$$

Here $P_{\mathrm{Ker}\, P P_n P}$ denotes the orthogonal projection onto $\mathrm{Ker}\, P P_n P$.

We can find $r > 0$ such that $B_r(0) \setminus \{0\}$ is in the resolvent set of $P P_n P$.

Then

$$
\begin{aligned}
P &= 1 - P_{\mathrm{Ker}\, P P_n P} \\
&= 1 - \frac{1}{2\pi i} \int_{|\lambda|=r} (\lambda - P P_n P)^{-1} d\lambda \\
&= \frac{1}{2\pi i} \int_{|\lambda|=r} \left(\lambda^{-1} - (\lambda - P P_n P)^{-1} \right) d\lambda \\
&= -\frac{1}{2\pi i} P P_n P \int_{|\lambda|=r} \lambda^{-1} (\lambda - P P_n P)^{-1} d\lambda \ .
\end{aligned}
$$

The integral kernel of $P P_n P$ fulfills the conditions of Prop. 5.3.1. From Cor. 5.3.2 we conclude that the spectrum of $P P_n P$ in $B(L^2(M, \mathcal{A}_i^l))$ is independent of $i \in \mathbb{N}_0$, thus

$$R := \int_{|\lambda|=r} \lambda^{-1} (\lambda - P P_n P)^{-1} d\lambda$$

is a bounded operator on $L^2(M, \mathcal{A}_i^l)$ for all $i \in \mathbb{N}_0$.

This and the equation $PR = PRP$ show that P is an integral operator with integral kernel

$$k_P(x, y) = -\frac{1}{2\pi i} \sum_{j=1}^{n} P e_j(x) (P R^* P e_j(y))^* \ .$$

The integral kernel is in $L^2(M \times M, M_l(\mathcal{A}_i))$ for all $i \in \mathbb{N}_0$ and is of the form we asserted.

(2) Let P_n as above with $\|(P_n - 1)P\| \leq \frac{1}{2}$ in $B(L^2(M, \mathcal{A}^l))$.

The operator $(1 + (P_n - 1)P)$ is invertible in $B(L^2(M, \mathcal{A}^l))$, hence by Prop. 5.3.1 it is invertible in $B(L^2(M, \mathcal{A}_i^l))$ for any $i \in \mathbb{N}_0$. From $P_n P = (1 + (P_n - 1)P)P$ it follows that

$$P_n P(L^2(M, \mathcal{A}_i^l)) \cong \operatorname{Ran} P(L^2(M, \mathcal{A}_i^l))$$

for all $i \in \mathbb{N}_0$. Furthermore $Q := (1 + (P_n - 1)P)P(1 + (P_n - 1)P)^{-1} \in B(L^2(M, \mathcal{A}_i^l))$ is a projection onto $P_n P(L^2(M, \mathcal{A}_i^l))$. Identify $P_n(L^2(M, \mathcal{A}_i^l))$ with $\mathcal{A}_\infty^n$ via the basis and let $Q' \in M_n(\mathcal{A}_\infty)$ be the restriction of Q to $P_n(L^2(M, \mathcal{A}_i^l))$. From $\operatorname{Ran}_\infty P \cong Q'(\mathcal{A}_\infty^n)$ it follows that $[\operatorname{Ran} P] = [Q']$ in $K_0(\mathcal{A})$ and $[\operatorname{Ran}_\infty P] = [Q']$ in $K_0(\mathcal{A}_\infty)$. This shows the assertion.

(3) Let P_n, Q and Q' be as in the proof of (2). In $H_*^{dR}(\mathcal{A}_\infty)$

$$
\begin{aligned}
\operatorname{ch}[\operatorname{Ran}_\infty P] &= \operatorname{ch}(Q') \\
&= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} \operatorname{tr}(Q' \, \mathrm{d}\, Q')^{2k} \\
&= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} \operatorname{Tr}((P_n Q P_n) \, \mathrm{d}(P_n Q P_n))^{2k} \\
&= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} \operatorname{Tr}(P_n Q)(\mathrm{d}\, P_n Q)^{2k} \\
&= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} \operatorname{Tr}(Q(\mathrm{d}\, Q)^{2k}) \, .
\end{aligned}
$$

Since

$$H : [0, 1] \to B(L^2(M, (\hat{\Omega}_{\leq \mu} \mathcal{A}_i)^l)) \, ,$$
$$H(t) = (1 + t(P_n - 1)P)P(1 + t(P_n - 1)P)^{-1}$$

is a differentiable path of finite projections with $H(0) = P$ and $H(1) = Q$, the difference $\operatorname{Tr}(Q \, \mathrm{d}\, Q)^{2k} - \operatorname{Tr}(P \, \mathrm{d}\, P)^{2k}$ is exact in $\hat{\Omega}_* \mathcal{A}_\infty / [\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty]_s$ by the previous lemma.

This shows the assertion. $\qquad\square$

5.4. Holomorphic semigroups

5.4.1. General facts. Let X be a Banach space.

In order to fix notation we collect some well-known facts about holomorphic semigroups on X. A general reference is [**RR**].

For $\delta \in\,]0, \pi]$ let $\Sigma_\delta := \{\lambda \in \mathbb{C}^* \mid |\arg \lambda| < \delta\}$.

Recall that a family $S : \Sigma_\delta \cup \{0\} \to B(X)$ is called a holomorphic semigroup if it is a semigroup, i.e. $S(0) = 1$ and $S(t + s) = S(t)S(s)$ for all $s, t \in \Sigma_\delta \cup \{0\}$, moreover if it is strongly continuous on $\Sigma_{\delta'} \cup \{0\}$ for all $\delta' < \delta$ and holomorphic on Σ_δ.

Given a holomorphic semigroup $S(t)$, the unbounded operator Z with $Zx := \frac{d}{dt}(S(t)x)|_{t=0}$ whenever defined is called the generator of Z. It is densely defined and closed. Any subset of its domain that is dense in X and invariant under the action of the semigroup is a core for Z.

Recall that a densely defined operator Z on X generates a holomorphic semi-group e^{tZ} if and only if there is $\omega \in \mathbb{R}$ such that $Z + \omega$ is δ-sectorial for some $\delta \in]0, \pi/2]$, i.e. if and only if $\Sigma_{\delta+\pi/2}$ is a subset of the resolvent set $\rho(Z + \omega)$ and for any ε with $0 < \varepsilon < \delta$ there is $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2}$

$$\|(Z + \omega - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} \ .$$

Then there is $C > 0$ such that for all $t > 0$

$$\|e^{tZ}\| \leq Ce^{-\omega t} \ .$$

LEMMA 5.4.1. *Let Z be a densely defined operator on X and let $\delta > 0$ be such that*

(i) $\Sigma_{\delta+\pi/2} \cup \{0\} \subset \rho(Z)$,
(ii) *for every $\alpha < \delta$ there are $c \in \mathbb{R}$ and $r > 0$ such that*

$$\|(Z + c - \lambda)^{-1}\| \leq \frac{C}{|\lambda|}$$

for $\lambda \in \Sigma_{\alpha+\pi/2}$ with $|\lambda| > r$ and $\lambda - c \in \rho(Z)$.

Then Z is δ-sectorial.

PROOF. Let $0 < \varepsilon < \delta$.

Let $\alpha \in (\varepsilon, \delta)$ and $R > r$ big enough such that any $\lambda \in \Sigma_{\varepsilon+\pi/2}$ with $|\lambda| > R$ satisfies $\lambda - c \in \Sigma_{\alpha+\pi/2}$. By assumption we find $C > 0$ such that for all $\lambda \in \Sigma_{\varepsilon+\pi/2}$ with $|\lambda| > R$

$$\begin{aligned}
\|(Z - \lambda)^{-1}\| &\leq& \frac{C}{|\lambda + c|} \\
&\leq& \left(\frac{C}{|\lambda|}\right)\left(\frac{|\lambda + c| + |c|}{|\lambda + c|}\right) \\
&\leq& \left(\frac{C}{|\lambda|}\right)\left(1 + \frac{|c|}{(R - |c|)}\right) \\
&\leq& \frac{C}{|\lambda|} \ .
\end{aligned}$$

Since $\overline{\Sigma_{\varepsilon+\pi/2}} \cap \{|z| \leq R\}$ is a compact subset of the resolvent set of Z, this implies the assertion.

$\square$

We have the following relation between the spectrum of a δ-sectorial operator Z and the behavior of the holomorphic semigroup e^{tZ} for $t \to \infty$.

PROPOSITION 5.4.2. *If Z is a δ-sectorial operator and there is $\omega > 0$ such that*

$$\{\operatorname{Re}\lambda > -\omega\} \subset \rho(Z) \ ,$$

then for any $\omega' < \omega$ there is $C > 0$ such that for all $t \geq 0$

$$\|e^{tZ}\| \leq Ce^{-\omega' t} \ .$$

PROOF. For any $\omega' < \omega$ there is $0 < \delta' \leq \delta$ such that $\Sigma_{\delta'+\pi/2} \cup \{0\} \subset \rho(Z+\omega')$. Now it follows from the previous lemma that $Z + \omega'$ is δ'-sectorial. $\square$

PROPOSITION 5.4.3. *If e^{tZ} is a strongly continuous semigroup with generator Z such that $\operatorname{Ran} e^{tZ} \subset \operatorname{dom} Z$ for all $t \in (0, \infty)$ and if there are $T > 0$ and $C > 0$ such that for $t \leq T$*

$$\|Ze^{tZ}\| \leq Ct^{-1} \ ,$$

then e^{tZ} extends to a holomorphic semigroup.

If the estimate holds for all $t > 0$, then the extension is bounded holomorphic.

PROOF. The assertion follows immediately from [**Da**], Th. 2.39. $\qquad\square$

Part (1) of the next proposition is known under the name Volterra development and the formula in part (2) is called Duhamel's formula:

PROPOSITION 5.4.4. *Let Z be the generator of a strongly continuous semigroup and let $M, \omega \in \mathbb{R}$ be such that $\|e^{tZ}\| \leq Me^{\omega t}$ for all $t \geq 0$.*

(1) *Let $R \in B(X)$. Then $Z + R$ is the generator of a strongly continuous semigroup and for all $t \geq 0$*

$$e^{t(Z+R)} = \sum_{n=0}^{\infty} t^n \int_{\Delta^n} e^{u_0 tZ} R e^{u_1 tZ} R \dots e^{u_n tZ} \ du_0 \dots du_n$$

with $\Delta^n = \{u_0 + \dots + u_n = 1; u_i \geq 0, \ i = 0, \dots, n\}$. Furthermore

$$\|e^{t(Z+R)}\| \leq Me^{(\omega + M\|R\|)t} \ .$$

(2) *Let $R_1, \dots, R_n \in B(X)$. For $t \geq 0$ the map*

$$\mathbb{C}^n \to B(X), \ (z_1, \dots, z_n) \mapsto e^{t(Z + z_1 R_1 + \dots + z_n R_n)}$$

is analytical and for $i = 0, \dots n$

$$\frac{d}{dz_i} e^{t(Z + z_1 R_1 + \dots + z_n R_n)} = \int_0^t e^{(t-s)(Z + z_1 R_1 + \dots + z_n R_n)} R_i e^{s(Z + z_1 R_1 + \dots + z_n R_n)} \ ds \ .$$

PROOF. (1) follows from [**Da**], Th. 3.1 and the proof of it.

(2) The analyticity follows from (1).

For the formula it is enough to consider $n = 1$. Let $R := R_1$.

For $z_0 \in \mathbb{C}$ by (1)

$$e^{(Z+zR)t} = \sum_{n=0}^{\infty} (z-z_0)^n t^n \int_{\Delta^n} e^{u_0 t(Z+z_0 R)} R e^{u_1 t(Z+z_0 R)} R \dots e^{u_n t(Z+z_0 R)} \ du_0 \dots du_n \ .$$

This implies that

$$\frac{d}{dz} e^{(Z+zR)t}\big|_{z_0} = t \int_0^1 e^{u_0 t(Z+z_0 R)} R e^{(1-u_0)t(Z+z_0 R)} du_0 \ .$$

$\qquad\square$

The following proposition is known in the literature as Duhamel's principle:

PROPOSITION 5.4.5. *Let Z be the generator of a strongly continuous semigroup on X. Let $u \in C^1([0, \infty), X)$ such that $u(t) \in \operatorname{dom} Z$ and $\frac{d}{dt}u(t) - Zu(t) \in \operatorname{dom} Z$ for all $t \in [0, \infty)$. Then*

$$e^{tZ}u(0) - u(t) = -\int_0^t e^{sZ}\Big(\frac{d}{dt} - Z\Big)u(t-s)ds \ .$$

PROOF. see [**Ta**], Appendix A, (9.37) and (9.38). $\qquad\square$

5.4.2. Square roots of generators and perturbations. Assume that D is a densely defined closed operator on a Banach space X with bounded inverse and such that $-D^2$ is δ-sectorial.

There are well-defined fractional powers $(D^2)^\alpha$ for $\alpha \in \mathbb{R}$ ([**RR**], §11.4.2). These are densely defined closed operators that coincide for $\alpha \in \mathbb{Z}$ with the usual powers and satisfy
$$(D^2)^{\alpha+\beta} f = (D^2)^\alpha (D^2)^\beta f$$
for all $\alpha, \beta \in \mathbb{R}$ and $f \in \operatorname{dom}(D^2)^\gamma$ with $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$. For $\alpha \leq 0$ the operator $(D^2)^\alpha$ is bounded and depends of α in a strongly continuous way.

In particular it follows that for $\alpha \geq 0$ the operator $(D^2)^{-\alpha}$ is a bounded inverse of $(D^2)^\alpha$.

Define $|D| := (D^2)^{\frac{1}{2}}$.

By [**Kat**], Th. 2, the operator $-|D|$ is $(\delta + \frac{\pi/2-\delta}{2})$-sectorial and can be expressed in terms of the resolvents of D^2.

Note that for every $n \in \mathbb{N}$ the domain of $|D|^n$ is a core of $|D|$ and $\operatorname{dom} D^n$ is a core of D.

LEMMA 5.4.6.   (1) *Let P be a closed densely defined operator on X and assume that P^2 is densely defined and $P^2|_M = P|_M$ for some dense subset M of $\operatorname{dom} P^2$.*

Then P is a bounded projection.

(2) *Let I be a closed densely defined operator on X and assume that I^2 is densely defined and that there is a dense subset M of $\operatorname{dom} I^2$ with $I^2|_M = 1|_M$.*

Then I is a bounded involution.

PROOF. (1) If $f \in M$, then $f = (1-P)f + Pf$ and $(1-P)Pf = P(1-P)f = 0$. We conclude that $M \subset \operatorname{Ker} P + \operatorname{Ker}(1-P) \subset \operatorname{dom} P$. If P is closed, then $\operatorname{Ker} P + \operatorname{Ker}(1-P)$ is closed, hence $\operatorname{Ker} P + \operatorname{Ker}(1-P) = X$, thus $\operatorname{dom} P = X$.

(2) The operator $P := \frac{1}{2}(1+I)$ is a closed projection on X in the sense of (1). Thus P is bounded. It follows that I is bounded as well.   □

PROPOSITION 5.4.7. *The closure of the operator $|D|^{-1}D : \operatorname{dom} D \to X$ is a bounded involution I on X and*

(1) $\operatorname{dom} D = \operatorname{dom} |D|$ *and* $I(\operatorname{dom} D) \subset \operatorname{dom} D$.
(2) $|D| = ID = DI$ *and* $D = I|D| = |D|I$.

PROOF. The operator D^{-1} commutes with the resolvents of D^2. It follows that $|D|^{-1}D^{-1} = D^{-1}|D|^{-1}$. Hence $|D|^{-1}(\operatorname{dom} D) \subset \operatorname{dom} D$ because of $\operatorname{dom} D = D^{-1}X$, so $\operatorname{dom} D^2 \subset \operatorname{dom}(|D|^{-1}D)^2$.

If $f \in \operatorname{dom} D$, then $|D|^{-1}Df = D|D|^{-1}f$. For $f \in \operatorname{dom} D^2$ it follows that $(|D|^{-1}D)^2 f = f$.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} D$ converging to zero. If $|D|^{-1}Df_n = D|D|^{-1}f_n$ converges, the limit is zero since D is closed. Hence $|D|^{-1}D$ is closable. By the previous lemma it extends to a bounded involution.

Since for $f \in \operatorname{dom} D^2$ we have that $DIf = |D|f$ and since DI is closed, the composition $IDI : \operatorname{dom} |D| \to X$ is well-defined and closed. It coincides with D on $\operatorname{dom} D^2$, hence it is a closed extension of D. It follows that $\operatorname{dom} D \subset \operatorname{dom} |D|$. The inclusion $\operatorname{dom} |D| \subset \operatorname{dom} D$ is shown analogously.

The equations are clear on $\operatorname{dom} D^2$, which is a core for D and $|D|$.   □

The operator $P := \frac{1}{2}(1 + I)$ with I as in the previous proposition is a bounded projection on X. By the proposition $P \operatorname{dom} D \subset \operatorname{dom} D$ and P commutes with D and $|D|$.

From $ID = |D|$ and $I|D| = D$ it follows that $PD = -P|D|$. Thus with respect to the decomposition $X = PX \oplus (1 - P)X$

$$D = \begin{pmatrix} PDP & 0 \\ 0 & -(1 - P)D(1 - P) \end{pmatrix}$$

and

$$|D| = \begin{pmatrix} PDP & 0 \\ 0 & (1 - P)D(1 - P) \end{pmatrix} .$$

By taking into account that $-|D|$ is $(\delta + \frac{\pi/2 - \delta}{2})$-sectorial it follows for the resolvent set of D:

PROPOSITION 5.4.8. *For $\lambda \in \mathbb{C}$*

$$\{\lambda, -\lambda\} \subset \rho(D) \Leftrightarrow \{\lambda, -\lambda\} \subset \rho(|D|) .$$

Thus if $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \Sigma_{\pi/2 + \delta}$, then $\lambda \in \rho(D)$.

Furthermore for every $\delta' < \delta$ there is $C > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\delta' + \pi/2}$

$$\|(D - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} .$$

COROLLARY 5.4.9. *Let $\omega \geq 0$ be such that there is $C > 0$ with $\|e^{-tD^2}\| \leq Ce^{-\omega t}$ for all $t \geq 0$.*

(1) *For every $\alpha \in \mathbb{R}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$*

$$\||D|^\alpha e^{-tD^2}\| \leq Ct^{-\alpha/2} e^{-\omega' t} .$$

(2) *For every $n \in \mathbb{N}$ and $\omega' < \omega$ there is $C > 0$ such that for all $t > 0$*

$$\|D^n e^{-tD^2}\| \leq Ct^{-n/2} e^{-\omega' t} .$$

PROOF. The first assertion is [**RR**], Lemma 11.36, and the second one follows from the first one by $D = I|D|$ and $DI = ID$. $\qquad \square$

PROPOSITION 5.4.10. *Let A be a bounded operator and let $\delta' < \delta$.*

(1) *There is $R > 0$ such that $D + A - \lambda$ has a bounded inverse if $|\lambda| > R$ and $-\lambda^2 \in \Sigma_{\delta' + \pi/2}$.*

(2) *There is $\omega > 0$ such that $-(D + A)^2 + \omega$ is δ'-sectorial.*

(3) *$D + A$ commutes with $e^{-t(D+A)^2}$.*

PROOF. By Prop. 5.4.8 there is $M > 0$ such that for all λ with $-\lambda^2 \in \Sigma_{\delta' + \pi/2}$

$$\|(D - \lambda)^{-1}\| \leq \frac{M}{|\lambda|}.$$

Hence the Neumann series

$$(D + A - \lambda)^{-1} = (D - \lambda)^{-1} \sum_{n=0}^{\infty} (A(D - \lambda)^{-1})^n$$

converges for $|\lambda| > M\|A\|$ and $-\lambda^2 \in \Sigma_{\delta' + \pi/2}$.

This shows (1).

If $|\lambda| > 2M\|A\|$ and $-\lambda^2 \in \Sigma_{\delta'+\pi/2}$, then

$$
\begin{aligned}
\|(D+A-\lambda)^{-1}\| &= \|\sum_{n=0}^{\infty}(D-\lambda)^{-1}\left(A(D-\lambda)^{-1}\right)^n\| \\
&\leq \sum_{n=0}^{\infty}\|A\|^n\|(D-\lambda)^{-1}\|^{n+1} \\
&\leq \sum_{n=0}^{\infty}\frac{M^{n+1}\|A\|^n}{|\lambda|^{n+1}} \\
&= \frac{M}{|\lambda|}(1-\frac{M\|A\|}{|\lambda|})^{-1} \\
&\leq \frac{2M}{|\lambda|} \ .
\end{aligned}
$$

Let $\mu \in \{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2}$. If $\lambda \in \mathbb{C}$ with $-\lambda^2 = \mu$, then $\lambda \in \rho(D+A)$, hence the resolvent

$$
(-(D+A)^2 - \mu)^{-1} = -(D+A-\lambda)^{-1}(D+A+\lambda)^{-1}
$$

exists and is bounded by

$$
\|(-(D+A)^2 - \mu)^{-1}\| \leq \frac{4M^2}{|\mu|} \ .
$$

There is $\omega > 4M^2\|A\|^2$ such that

$$
\Sigma_{\delta'+\pi/2} \cup \{0\} \subset \left(\{|\mu| > 4M^2\|A\|^2\} \cap \Sigma_{\delta'+\pi/2}\right) - \omega
$$

and thus

$$
\Sigma_{\delta'+\pi/2} \cup \{0\} \subset \rho\left(-(D+A)^2 + \omega\right) \ .
$$

Assertion (2) follows now from Lemma 5.4.1.

(3) follows from the fact that $e^{-t(D+A)^2}$ can be expressed in terms of the resolvents of $(D+A)^2$, which commute with $D+A$. $\square$

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